PI MU EPSILON JOURNAL

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Editorial correspondence, including manuscripts and news items, should be sent to Russell Euler, Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, MO 64468 (e-mail address, REuler@mail.nwmissouri.edu). Guidelines for authors are found on page 441 of the Volume 10, Number 5, Fall 1996 issue.

Problems for solution and solutions to problems should be sent directly to Clayton W. Dodge, Mathematics Department, 5752 Neville Hall, University of Maine, Orono, Maine 04469.

Correspondence regarding subscriptions, back issues or changes of address should be addressed to Joan Weiss, Department of Mathematics and Computer Science, Fairfield University, Fairfield, CT 06430.

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Seeing Is Not Always Believing

Loi Nguyen (student)
Tu Tran (student)
University of North Florida

Many years ago, Loi was asked the geometric conjecture below. He could not determine its validity until he took a geometry course at the University of North Florida.

Conjecture: Let MN and PQ be two perpendicular diameters in a circle with center O, and A, B be two points on $\overline{MO}$ on $\overline{ON}$ such that MA = OB. If line BQ intersects the circle at C, then $\angle$ CAQ is a right angle.

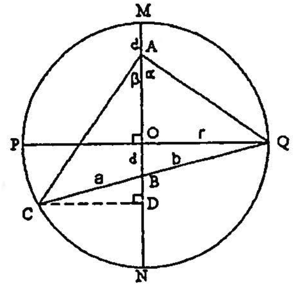

FIGURE 1

Let us consider two special cases:

1. A coincides with M. Then C coincides with P, and $\angle$ PMQ is a right angle.
2. A coincides with O. Then C coincides with N, and $\angle$ NOQ is a right angle.

In other cases, if a protractor is used to measure the angle, it always appears to have measure 90°.

Surprisingly, we are going to prove that the conclusion in the conjecture above is not correct although it seems true. Our method combines geometry, trigonometry, and calculus together. The correct conclusion of the conjecture

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should be: $\angle$ CAQ is an angle with measure between 90° and approximately 88.21°. Since 88.21° is so close to 90°, the naked eye can hardly find the difference.

Now let us turn to the proof. Let MA = OB = d, the radius of the circle be r, CB = a, BQ = b, $\angle$ CAB = β, and $\angle$ BAQ = α. Since $\overline{CQ}$, $\overline{MN}$ are two chords of a circle,

$$CB \cdot BQ = MB \cdot BN,$$

or $ab = (r + d)(r - d) = r^2 - d^2$.

Since $\triangle$ BOQ is a right triangle, we have

$$b^2 = r^2 + d^2.$$

Hence $a/b = (r^2 - d^2)/(r^2 + d^2)$.

Find the point D on $\overline{MN}$ so that $\overline{CD} \perp \overline{MN}$. Since $\triangle BDC \sim \triangle BOQ$, we know

$$BD/a = d/b, \quad CD/a = r/b.$$

Hence,

$$BD = d(a/b) = d(r^2 - d^2)/(r^2 + d^2),$$
$$CD = r(a/b) = r(r^2 - d^2)/(r^2 + d^2).$$

It is easily seen that

$$\tan\alpha = r/(r - d) = 1/(1 - (d/r)),$$

and

$$\begin{aligned}\tan\beta &= CD/AD = CD/(r + BD)\\ &= [r(r^2 - d^2)/(r^2 + d^2)]/[r + d(r^2 - d^2)/(r^2 + d^2)]\\ &= r(r^2 - d^2)/[r(r^2 + d^2) + d(r^2 - d^2)]\\ &= [1 - (d/r)^2] / [1 + (d/r) + (d/r)^2 - (d/r)^3].\end{aligned}$$

Let $x = d/r$. Then $0 \le x \le 1$, and

$$\tan\alpha = 1/(1 - x),$$
$$\tan\beta = (1 - x^2)/(1 + x + x^2 - x^3),$$
$$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta)$$
$$= [1/(1 - x) + (1 - x^2)/(1 + x + x^2 - x^3)]/[1 - [1/(1 - x)][(1 - x^2)/(1 + x + x^2 - x^3)]]$$
$$= 2/(x^2(1 - x)^2).$$

Let $f(x) = \tan(\alpha + \beta) = 2/(x^2(1 - x)^2)$. Then $f'(x) = -4(1 - 2x)/(x - x^2)^3$.

If $0 < x < 1/2$, then $f'(x) < 0$; hence, $f(x)$ is decreasing on (0, 1/2). If $1/2 < x < 1$, then $f'(x) > 0$; hence, $f(x)$ is increasing on (1/2, 1). Therefore $x = 1/2$ results in a minimum value of $f(x)$.

It is easily seen that

$$f(1/2) = 2/[(1/4)(1/4)] = 32.$$

The minimum value of the angle $\angle$ CAQ is:

$$\arctan f(1/2) = \arctan(\alpha + \beta) = \arctan 32 \approx 88.210089°.$$

The following are some values from which one can see that $\angle$ CAQ is very close to 90°.

$x = 1/3$	$f(1/3) = 81/2$	$\arctan(81/2) \approx 88.5855770°$
$x = 1/4$	$f(1/4) = 512/9$	$\arctan(512/9) \approx 88.9929510°$
$x = 1/5$	$f(1/5) = 625/8$	$\arctan(625/8) \approx 89.266654°$.

Acknowledgment. We thank Professor Jingcheng Tong and the referee for their guidance in preparing this article.

Integral Functions Whose Right Derivatives Are Average Values Of Periodic Functions

Paul Fishback
Nicholas Ceglarek (student)
Tobias Moleski (student)
Grand Valley State University

One of the most beautiful results in mathematics is the first fundamental theorem of calculus. Recently, several articles ([1], [2], [3], and [4]) have focused on an interesting class of highly-oscillatory functions that do not satisfy this theorem's hypotheses. To be in this class, a function f must

(a) be defined and bounded on the interval $[0, 1]$;

(b) be continuous on $(0, 1]$;

(c) have the property that $\lim_{x\to 0^+} f(x)$ does not exist.

Because f is bounded on $[0, 1]$, where it has a single discontinuity, the Riemann integral $\int_0^x f(t)dt$ exists for every $x \in [0, 1]$. However, whether this integral is right-differentiable at the origin, i.e. whether

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x f(t)dt$$

exists, cannot be answered on the basis of properties (a) - (c) alone.

Steve Ricci, for example, has shown ([4]) that if

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } 0 < x \le 1 \\ 1 & \text{if } x = 0 \end{cases}, \tag{1}$$

then

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x f(t)dt = 0.$$

John Klippert ([3]), on the other hand, has shown that when $\ln x$ replaces $\frac{1}{x}$

in (1), $\lim_{x\to 0^+} \frac{1}{x}\int_0^x f(t)dt$ does not exist. The fact that these outcomes are different has been explained in terms of the different growth rates of the derivatives of $1/x$ and $\ln x$ near the origin ([1]).

Other examples of functions satisfying (a) - (c) can be created simply by replacing the sine function in (1) by some other continuous, periodic function g. For some such choices of g, e.g. $g = \cos$, an easy adaptation of Ricci's argument establishes that

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x f(t)dt$$

exists.

For other choices, however, the outcome is not so clear. One such example is $g = |\sin|$. In this case experimentation with a computer algebra system suggests that

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x f(t)dt \approx .63662.$$

The following theorem, which uses integration by parts and properties of infinite series to establish a general result for all continuous, periodic functions g, demonstrates that this limit is $2/\pi$ the average value of $|\sin|$ over one period.

Theorem: Suppose g is a continuous periodic function on the real line, whose period is denoted by τ. Then

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x g\left(\frac{1}{t}\right)dt = \frac{1}{\tau}\int_0^\tau g(s)ds.$$

Proof: By a simple change of variable, the limit in question may be rewritten as

$$\lim_{\lambda\to\infty} \lambda\int_\lambda^\infty \frac{g(s)}{s^2}ds.$$

Consider first the special case $\lambda = N\tau$, where N is a positive integer. By the periodicity of g,

$$\lambda\int_\lambda^\infty \frac{g(s)}{s^2}ds = N\tau \sum_{j=N}^{\infty} \int_{j\tau}^{(j+1)\tau} \frac{g(s)}{s^2}ds$$

$$= N\tau \sum_{j=N}^{\infty} \int_0^{\tau} \frac{g(s)}{(s+j\tau)^2} ds.$$

Integration by parts applied to the integral inside this series, together with the continuity of g, yields

$$\int_0^{\tau} \frac{g(s)}{(s+j\tau)^2} ds = \frac{1}{\tau^2(1+j)^2} \int_0^{\tau} g(s)ds + \int_0^{\tau} \frac{2}{(s+j\tau)^3} \left(\int_0^{s} g(\tau)d\tau \right) ds$$

$$= \frac{1}{\tau^2(1+j)^2} \int_0^{\tau} g(s)ds + O\left(\frac{1}{j^3}\right).$$

Hence,

$$\lim_{\lambda\to\infty} \lambda \int_{\lambda}^{\infty} \frac{g(s)}{s^2} ds = \lim_{N\to\infty} N\tau \sum_{j=N}^{\infty} \int_0^{\tau} \frac{g(s)}{(s+j\tau)^2} ds$$

$$= \lim_{N\to\infty} N\tau \sum_{j=N}^{\infty} \left(\frac{1}{\tau^2(1+j)^2} \int_0^{\tau} g(s)ds + O\left(\frac{1}{j^3}\right) \right)$$

$$= \lim_{N\to\infty} N \left(\sum_{j=N}^{\infty} \frac{1}{(1+j)^2} \left(\frac{1}{\tau} \int_0^{\tau} g(s)ds \right) + O\left(\frac{1}{j^3}\right) \right).$$

Since $\lim_{N\to\infty} N \sum_{j=N}^{\infty} \frac{1}{(1+j)^2} = 1$ and $\lim_{N\to\infty} N \sum_{j=N}^{\infty} \frac{1}{j^3} = 0$, the result follows for this special case.

For the general case, denote $N_\lambda = \left[\frac{\lambda}{\tau}\right]$, the greatest integer less than or equal to $\frac{\lambda}{\tau}$. Then

$$\lambda \int_{\lambda}^{\infty} \frac{g(s)}{s^2} ds = \lambda \left(\int_{\lambda}^{\infty} \frac{g(s)}{s^2} ds - \int_{N_\lambda\tau}^{\infty} \frac{g(s)}{s^2} ds + \int_{N_\lambda\tau}^{\infty} \frac{g(s)}{s^2} ds. \right)$$

$$= -\lambda \int_{N_\lambda\tau}^{\lambda} \frac{g(s)}{s^2} ds + \lambda \int_{N_\lambda\tau}^{\infty} \frac{g(s)}{s^2} ds.$$

If M denotes the maximum of $|g|$, then the first of these two integrals is bounded in absolute value by

$$\lambda M \left(\frac{1}{N_\lambda\tau} - \frac{1}{\lambda} \right) = M \left(\frac{\lambda}{N_\lambda\tau} - 1 \right).$$

Since $\lim_{x\to\infty} \frac{x}{[x]} = 1, \lim_{\lambda\to\infty} \frac{\lambda}{N_\lambda\tau} = 1$, which implies that the first integral approaches zero as λ approaches infinity. The second integral is the same as $\frac{\lambda}{N_\lambda\tau} \left(N_\lambda\tau \int_{N_\lambda\tau}^{\infty} \frac{g(s)}{s^2} ds \right)$, which, by the special case, approaches $\frac{1}{\tau} \int_0^{\tau} g(s)ds$ as λ approaches infinity. This completes the proof. ◊

Of course, if g is merely a continuous, bounded (but not necessarily periodic) function defined on the real line, then

$$f(x) = \begin{cases} g\left(\frac{1}{x}\right) & \text{if } 0 < x \le 1 \\ 1 & \text{if } x = 0 \end{cases}$$

satisfies conditions (a) - (c) as well. Does $\lim_{x\to 0^+} \frac{1}{x} \int_0^x f(t)dt$ always exist in this more general case?

The answer to this question is no. To see why this is so, choose any sequence of positive numbers $\{\alpha_n\}$ that increases to infinity quickly enough so that

$$\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 0.$$

Now construct a bounded, continuous function g that satisfies the following properties:

(1) $g(x) = 0$ if $x \in [0, \alpha_1]$ or if $x \in [\alpha_n, \alpha_{n+1}]$ for some even value of n;
(2) $0 < |g(x)| \le 1$ if $x \in (\alpha_n, \alpha_{n+1})$ for some odd value of n;
(3) $g(x) = g(-x)$, and g is continuous on $(-\infty, \infty)$;
(4) if n is odd,

$$\left|\int_{\alpha_n}^{\alpha_{n+1}}\frac{g(s)}{s^2}ds - \int_{\alpha_n}^{\alpha_{n+1}}\frac{g(s)}{s^2}\right| < \frac{1}{\alpha_{n+1}}.$$

Such a bounded, continuous function can be constructed by using a piecewise-linear function. For example, g might have a graph that looks like Figure 1:

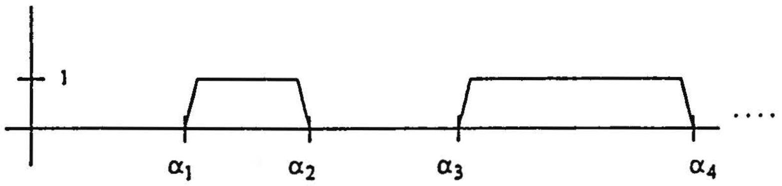

FIGURE 1

Now consider

$$\alpha_n\int_{\alpha_n}^{\infty}\frac{g(s)}{s^2}ds = \alpha_n\int_{\alpha_n}^{\alpha_{n+1}}\frac{g(s)}{s^2}ds + \alpha_n\int_{\alpha_{n+1}}^{\infty}\frac{g(s)}{s^2}ds.$$

Since $|g| \le 1$, the second of these two integrals is bounded by $\frac{\alpha_n}{\alpha_{n+1}}$, and hence goes to zero as $n \to \infty$.

The first integral tends to two different limits depending upon whether n is even or odd. If n is even, the integral equals zero by (1). If n is odd, the same integral can be rewritten as

$$\alpha_n\int_{\alpha_n}^{\alpha_{n+1}}\frac{g(s)}{s^2}ds$$

$$= \alpha_n\left(\int_{\alpha_n}^{\alpha_{n+1}}\frac{g(s)}{s^2}ds - \int_{\alpha_n}^{\alpha_{n+1}}\frac{1}{s^2}ds\right) + \alpha_n\int_{\alpha_n}^{\alpha_{n+1}}\frac{1}{s^2}ds$$

$$= \alpha_n\left(\int_{\alpha_n}^{\alpha_{n+1}}\frac{g(s)}{s^2}ds - \int_{\alpha_n}^{\alpha_{n+1}}\frac{1}{s^2}ds\right) + \left(1 - \frac{\alpha_n}{\alpha_{n+1}}\right).$$

By (4)

$$\alpha_n\left|\int_{\alpha_n}^{\alpha_{n+1}}\frac{g(s)}{s^2}ds - \int_{\alpha_n}^{\alpha_{n+1}}\frac{1}{s^2}\right| < \frac{\alpha_n}{\alpha_{n+1}}.$$

Using again the fact that $\frac{\alpha_n}{\alpha_{n+1}}$ tends to zero as $n \to \infty$, we obtain

$$\lim_{n\to\infty,\ n \text{ odd}} \alpha_n\int_{\alpha_n}^{\alpha_{n+1}}\frac{g(s)}{s^2}ds = 1.$$

Hence

$$\lim_{x\to 0^+}\frac{1}{x}\int_0^x g\left(\frac{1}{t}\right)dt = \lim_{\lambda\to\infty}\int_{\lambda}^{\infty}\frac{g(s)}{s^2}ds$$

does not exist for this particular bounded, continuous function g.

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On Jacobians in Multiple Integrals

Prem N. Bajaj
Wichita State University

In change of variables for double or triple integrals, the Jacobian J of the transformation is, generally, either non-negative or non-positive on the entire region of integration. Consequently students do not give consideration to the absolute value sign present in the change of variables formula (except perhaps to make J positive if J is non-positive on the region of integration). However, the Jacobian J may be both positive and negative on the region of integration. In such cases the integral needs to be evaluated separately on the regions where J is positive and negative. It is one of such cases that we consider.

We are interested in evaluation of the integral

$$I=\iint_R \sqrt{x^2+y^2}\, dA \text{ where } R \text{ is the region } \{(x,y) : (x-1)^2+y^2 \le 1\}.$$

Consider the following "solution". In polar coordinates, the region can be described as

$$\{(r,\theta) : r = 2\cos\theta\ ,\ 0 \le \theta \le \pi\}.$$

Using the transformation $x = r\cos\theta$, $y = r\sin\theta$ we obtain

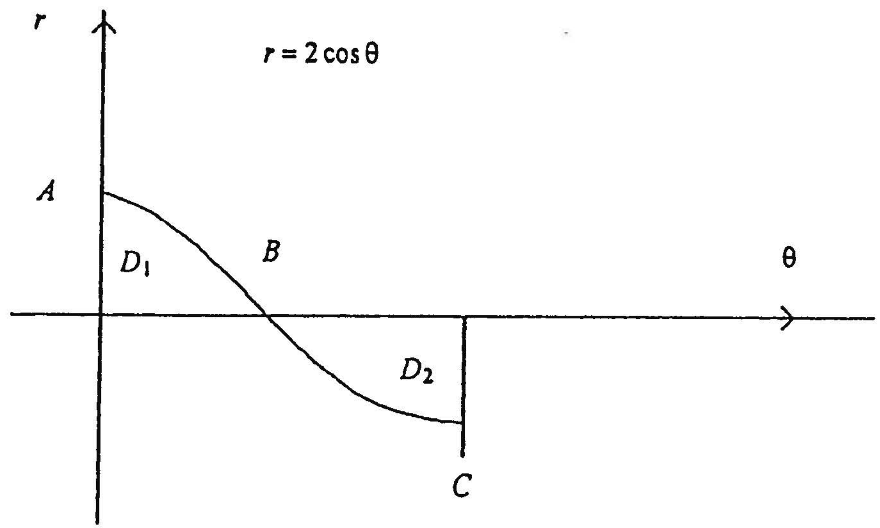
$$I = \int_0^{\pi}\int_0^{2\cos\theta} r\cdot r\ dr d\theta = \int_0^{\pi} \frac{8}{3}\cos^3\theta\ d\theta = 0.$$

Since the integrand is positive for all $(x,y) \neq (0,0)$ and f is continuous on R, the integral cannot vanish. What went wrong?

The region R in the x-y plane transforms to the region D in plane θ-r plane where Jacobian, J, of the transformation takes both positive and negative signs. Indeed r is positive for $0 \le \theta < \pi/2$ and negative for $\pi/2 < \theta \le \pi$. The correct integrand, in the preceding double integral, according to the change of variables formula, is given by $r\cdot|r|$; consequently the integral is to be evaluated separately on regions D_1 and D_2 (see figure). We leave it to the reader to check that the value of the integral is 32/9.

Acknowledgment. The author is thankful to the referee for helpful comments

on the earlier version of this note.



Figure

Here points A, B and C correspond, respectively, to $\theta = 0, \frac{\pi}{2}$ and π.

residual term $2c(\log x)^{1/2}$ is denoted by $\Delta(x)$. The author is indebted to Dr. Peter G. Anderson of the Rochester Institute of Technology for the computations of $f(x)$; these were subsequently checked by the anonymous referee using Mathematica and found to contain very minor discrepancies. The corrected values have been incorporated in Table 1.

TABLE 1

m	n	$\pi(x)$	li(x)	$f(x)$	$\Delta(x)$
3	6	168	178	177	13
4	9	1,229	1,246	1,247	15
5	11	9,592	9,630	9,630	17
6	13	78,498	78,628	78,628	19
7	16	664,579	664,918	664,919	20
8	18	5,761,455	5,762,209	5,762,210	22
9	20	50,847,534	50,849,235	50,849,235	23
10	23	455,052,511	455,055,615	455,055,615	24
11	25	4,118,054,813	4,118,066,401	4,118,066,401	25
12	27	37,607,912,018	37,607,950,281	37,607,950,281	26
13	29	346,065,536,839	346,065,645,810	346,065,645,810	27
14	32	3,204,941,750,802	3,204,942,065,692	3,204,942,065,692	28
15	34	29,844,570,422,669	29,844,571,475,288	29,844,571,475,288	29
16	36	279,238,341,033,925	279,238,344,248,557	279,238,344,248,557	30
17	39	2,623,557,157,654,233	2,623,557,165,610,822	2,623,557,165,610,822	31
18	41	24,739,954,287,740,860	24,739,954,309,690,415	24,739,954,309,690,415	32

To the best of the author's knowledge, the greatest value of x for which $\pi(x)$ and li(x) have *both* been computed is 10^{18}, which is the limit of the table above. These values were reported by M. Deleglise and J. Rivat [1] in 1996, who, in turn, refer to the 1985 paper [3] by J.C. Lagarias, V.S. Miller and A.M. Odlyzko.

We see that for most practical purposes, we may ignore the error term $\Delta(x)$ in (7) and simplify the relation to the following:

$$\text{li}(x) = f(x) + O((\log x)^{1/2}). \qquad (8)$$

As we can see, $f(x)$ as given by the formula in (4) approximates li(x) with an astonishing degree of accuracy, at least for the range of values examined.

We note that we cannot extend the upper limit in the series defining $f(x)$ to infinity, since we would then be dealing with a divergent series. The upper limit [log x] has been fortuitously selected to make the successive terms of the series *decrease* with ascending k (as we can satisfy ourselves by applying a ratio test). It is safe to say that $f(x)$ is a definite improvement numerically over the usual estimates implied by the Prime Number Theorem.

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On the Chromatic Number of the Middle Graph of a Graph

Masakazu Nihei
Fujishiro High School

Introduction. In recent years the number of students learning graph theory has increased, because there are applications of graph theory to some areas of computer science, chemistry, physics, economics, etc., and graph theory is also intimately related to many branches of pure mathematics; for example, group theory, matrix theory, topology, etc. The coloring of graphs is one of the most interesting branches in graph theory yet, and moreover there are some applications concerning the graph colorings. It therefore will be worth while studying the graph coloring.

Definitions and notations. A *graph* G is a finite nonempty set of objects called *vertices* (the singular is vertex) together with a (possibly empty) set of unordered pairs of distinct vertices of called *edges*. We denote the set of vertices and edges of a graph G by $V(G)$ and $E(G)$, respectively. When $E(G) = \emptyset$, G is called an *empty graph*.

Let G be a graph. The edge $e = \{u, v\}$ is said to join the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent* vertices, while u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are adjacent edges. The degree of a vertex v of G is the number of edges of G incident with v. The degree of a vertex v in G is denoted $\deg_G v$ or simply $\deg v$.

An assignment of colors to the vertices of a graph G, one color to each vertex, so that adjacent vertices are assigned different colors is called a *coloring* of G; a coloring in which n colors are used is an *n-coloring*. A graph of G is n-colorable if there exists an m-coloring of for some $m \leq n$. The minimum n for which a graph G is n-colorable is called the *vertex chromatic number* of G, and is denoted by $\chi(G)$. When $\chi(G) = n$, G is called *n-chromatic*. For example, a graph in which every two vertices are adjacent is called a *complete graph*. The complete graph with n vertices is denoted by K_n. Then $\chi(K_n) = n$.

The *middle graph* $M(G)$ of a graph G is the graph obtained from G by

inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G (see Figure 1).

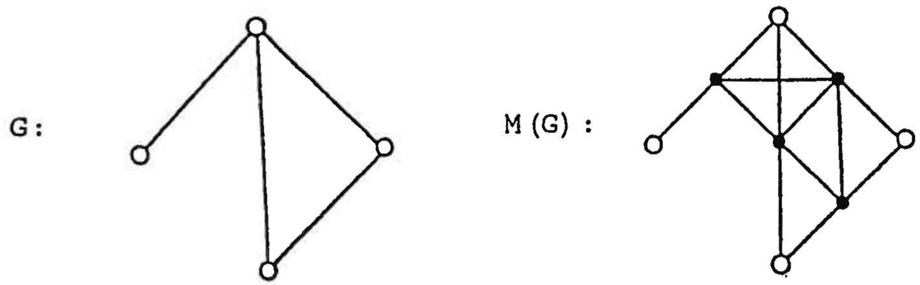

Figure 1

Although the middle of a graph has been investigated by several authors ([3], [4]) since it was first discovered in 1973, the chromatic number of the middle graph of a graph has not been studied. Now, in this note, we shall give the chromatic number of the middle graph $M(G)$ of a graph G. (Notations and definitions not given here can be found in [1]).

The edge chromatic number of the endline graph of a graph. In this section, we will prepare the theorem and lemma for the proof of our main results.

An assignment of colors to the edges of a graph G, one color to each edge, so that adjacent edges are assigned different colors is called an *edge coloring* of G; an edge coloring in which n colors are used is an *n-edge coloring*. A graph G is n-edge colorable if there exists an m-edge coloring for some $m \leq n$. The minimum n for which a graph G is n-edge colorable is called the *edge chromatic number* (or *chromatic index*) of G, and is denoted by $\chi_1(G)$. If $\chi_1(G) = k$, we say that G is *k-edge-chromatic*. For example (see [1, p. 288] or [2, p. 93]),

$$\chi_1(K_n) = \begin{cases} n & \text{if } n \text{ is odd } (n \neq 1) \\ n-1 & \text{if } n \text{ is even} \end{cases} \qquad (1)$$

Let $\Delta(G)$ be the maximum degree among the vertices of G. Then $\Delta(G)$ is clearly a lower bound for $\chi_1(G)$. The following fundamental result on edge colorings was given by Vizing (see [1, Theorem 10.11 (p. 286)]).

Theorem 1 (Vizing). Let G be any graph. Then

$$\chi_1(G) = \Delta \text{ or } \Delta + 1,$$

where Δ is the maximum degree of G.

Graphs for which $\chi_1(G) = \Delta$ are called *class* 1, and those for which $\chi_1(G) = \Delta + 1$ are called *class* 2. It is unknown in general which graphs belong to which class.

The *line graph* $L(G)$ of a nonempty graph G is the graph whose vertices are in one-one correspondence with the edges of G, two vertices $L(G)$ being adjacent if and only if the corresponding edges are adjacent. Then, from the definitions, it is immediate that $\chi_1(G) = \chi(L(G))$. Let $V(G) = \{v_1, v_2, \cdots, v_p\}$. To G, we add p new vertices and p edges $\{u_i, v_i\}$ $(i = 1, 2, \cdots, p)$, where u_i's are different from any vertex of G and from each other. Then we obtain a new graph with $2p$ vertices. Let us denote this graph by G^+ and call it the *endline graph* of G. We also call an edge $\{u_i, v_i\}$ an endline of G. Then we have the following result:

Lemma 1. Let G be any graph. Then

$$\chi_1(G^+) = \Delta + 1,$$

where Δ is the maximum degree of G.

Proof. We may consider the following two cases from Theorem 1.

Case 1. G is of class one. In this case, we are given a Δ-edge coloring of G. Now, we choose a color c different from Δ colors, and assign it to all the endlines of G. Then we have a $(\Delta + 1)$-edge coloring of the endline graph G^+. On the other hand, we have $\chi_1(G^+) \geq \Delta + 1$ since $(\Delta + 1)$ is the maximum degree of G^+. We hence obtain $\chi_1(G^+) = \Delta + 1$.

Case 2. G is of class two. In this case, we are given a $(\Delta + 1)$-edge coloring of G. Let v be any vertex of G. Then there is at least one color missing the edges incident with v since Δ is the maximum degree of G. Choosing one color from them, we assign it to the endline of v. Then we have a $(\Delta + 1)$-edge coloring of G^+. This implies $\chi_1(G^+) = \Delta + 1$. Therefore, we obtain the desired result. ■

Main results. From the definitions of the endline graph and the middle graph of a graph, it is already known ([3]) that the following result holds.

Theorem 2 ([3]). Let G be any graph. Then

$$L(G^+) \cong M(G),$$

where the symbol '$\cong$' means isomorphism.

By means of Lemma 1 and Theorem 2, we arrive at our main result.

Theorem 3. Let G be any graph. Then

$$\chi(M(G)) = \Delta + 1,$$

where Δ is the maximum degree of G.

Proof. When $E(G) = \emptyset$, Theorem 3 is clear. Hence we may assume $E(G) \neq \emptyset$. Then we can easily check that the following equality holds:

$$\chi(M(G)) = \chi(L(G^+)) = \chi_1(G^+) = \Delta + 1.$$

This completes the proof. ■

By using a similar argument, we can simply give the *total chromatic number* of the complete graph K_n. The total chromatic number $\chi_2(G)$ of G is the least number of colors needed to color vertices and edges of G so that no adjacent vertices, adjacent edges, or incident edges and vertices, are assigned the same color. For example, $\chi_2(K_4) = 5$ (see Figure 2).

K_4 :

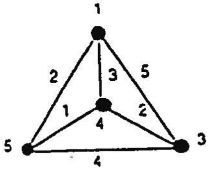

Figure 2

The *total graph* $T(G)$ of a graph G is that graph whose set of vertices is the union of the set of vertices and of the set of edges of G, with two vertices in $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident.

From the preceeding definition, it is immediate that

$$\chi_2(G) = \chi(T(G)). \qquad (2)$$

By (1) and (2), we can give another simple proof of the following well-known result ([5]):

$$\chi_2(K_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even.} \end{cases} \qquad (3)$$

In fact,

$$\chi_2(K_n) = \chi(T(K_n)) = \chi(L(K_{n+1})) = \chi_1(K_{n+1}) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even.} \end{cases} \blacksquare$$

In 1965, M. Behzad ([6]) has conjectured that, for any graph G, $\chi_2(G) \le \Delta + 2$. This conjecture is open yet.

Acknowledgement. The author would like to thank the referee for helpful suggestions.

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Involutions in Permutations

Subhash C. Saxena
Coastal Carolina University

Permutations are one-one mappings of a non-empty set onto itself, and play important roles in the study of group theory and geometry. In this brief note, we focus on special types of permutations that are not commonly discussed in textbooks. They are called involutions or involutory permutations. The concept of involution is ordinarily introduced in geometry textbooks [1, 2, 4, 5] in the context of transformations. In abstract algebra, it is obscurely mentioned, such as in an exercise in [3]. However, the definition given in [3] differs from the one given here which is similar to those in geometry textbooks. In this paper, a recursive relation is developed and two theorems are proved.

Definition 1: A permutation p on a non-empty set is called an involution if it is not the identity, but its square (p^2) is the identity.

In the lexicon of group theory, an involution is of order two; in the language of geometry, it is of period two. In terms of electronic gadgetry, an involution may be considered as a "toggle switch."

It follows from the definition that an involution on a set is a transposition or is a product of disjoint transpositions. (A transposition is a permutation which interchanges exactly two elements.) Thus, in an involution, each element is either transposed with another element or remains fixed.

In this note, we shall follow the usual convention of cyclic notation for a permutation. For example, (142) on the set {1, 2, 3, 4} of four elements is a transformation described by $1 \to 4$; $4 \to 2$; $2 \to 1$, and 3 remains fixed. (This permutation is not an involution.)

Examples of involutions on this set of four elements are (14), (13)(24); among others. However, (123) or a permutation containing (123) as a cycle cannot be an involution since (123)(123) = (132).

In this paper, the set of all involutions obtained from a group S_n of permutations on n elements is denoted by I_n. For example, the only involution in the group S_2 of permutations on two elements {1, 2} is denoted by $I_2 = \{(12)\}$. It consists of a single transposition. Obviously, there is no involution on S_1. For S_3, we have, $I_3 = \{(12),(13),(23)\}$; which consists of three distinct transpositions.

There is a recursive relationship among permutations, which we prove in the following theorem.

Theorem 1: For n > 1, $|I_{n+1}| = |I_n| + n\,|I_{n-1}| + n$, where $|I_n|$ denotes the cardinal number of I_n.

Proof: Given I_n on a set, a new (n + 1)th element when appended to an n-set can produce involutions in S_{n+1} in the following mutually exclusive ways: (i) For each involution of I_n, construct a new permutation on the (n + 1)-set in such a manner that the (n + 1)th element remains fixed, and the given involution is not disturbed. This permutation is still an involution in S_{n+1}. The number of involutions in such a process is $|I_n|$. This procedure does not produce any additional involution. (ii)Transpose the (n + 1)th element with each of the elements in the original n-set and combine it with each involution on the remaining (n - 1) elements. The number of involutions in this process is $n\,|I_{n-1}|$. (iii) Transpose the (n + 1)th element with each element of the old n-set in turn, and keep all the remaining elements invariant. Each such transposition will occur n times.

Adding the results in (i), (ii), and (iii) we obtain

$$|I_{n+1}| = |I_n| + n\,|I_{n-1}| + n;$$

which proves the theorem, since all the involutions in S_{n+1} are covered in this procedure, and none is duplicated. <>

By the formula in the theorem $|I_4| = 9$, and $|I_5| = 25$. For illustrations, we obtain I_4 and I_5.

As mentioned earlier, I_3 = {(1 2), (1 3), (2 3)}. For I_4, first we retain the involutions I_3 . Next, the process (ii) described in the proof will produce the following additional involutions from I_2: (1 4)(2 3), (2 4)(1 3), and (3 4)(1 2); whereas the procedure (iii) will yield (1 4), (2 4), and (3 4).

Thus, I_4 = {(1 2), (1 3), (2 3); (1 4)(2 3), (2 4)(1 3), (3 4)(1 2); (1 4), (2 4), (3 4)}.

Similarly, I_5 = {(1 2), (1 3), (2 3), (1 4)(2 3), (2 4)(1 3), (3 4)(1 2), (1 4), (2 4), (3 4); (1 5)(2 3), (1 5)(2 4), (1 5)(3 4), (2 5)(1 3), (2 5)(1 4), (2 5)(3 4), (3 5)(1 2), (3 5)(1 4), (3 5)(2 4), (4 5)(1 2), (4 5)(1 3), (4 5)(2 3); (1 5), (2 5), (3 5), (4 5)}.

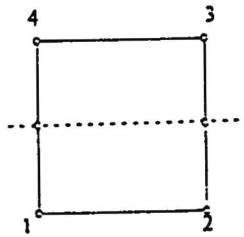
In view of the fact that each permutation can be expressed as a product of transpositons, and every transposition is itself an involution, the following result is established.

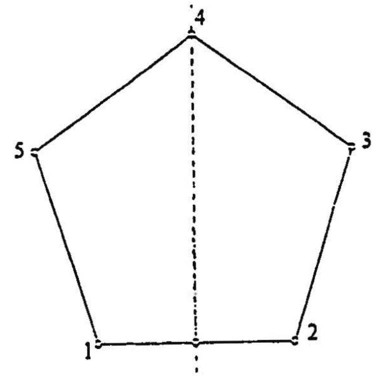
Theorem 2: Every permutation on an n-set is a product of involutions in S_n.

Thus, we can regard involutions as building blocks of permutations, in the same manner as line reflections on a plane are building blocks of all Euclidean transformations. This building block has a wider base than that of transpositions.

In fact, some involutions on a given n-set can be interpreted as line reflections on a regular n-gon formed by the vertices identified by the n-set in a specific order.

For examples, (a) (1 4)(2 3) is a line reflection for the square 1234 through the common perpendicular bisector of the opposite sides 14 and 23; (b) (3 5)(1 2) is a reflection through one of the five axes of symmetries, this one passing through the vertex 4.





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Geometric Interpretations of the Continuous Mean

Arvind Shah
Xin-Min Zhang
University of South Alabama

Introduction. The theory of means and their inequalitites are very basic and important in many fields including mathematics, statistics, physics, and economics. Motivated by different concerns, there are numerous ways to introduce mean values. For two given positive real numbers $a \leq b$, a mean is an intermediate value between a and b. The following three are the best known and frequently used means:

1. The arithmetic mean $m_1 = (a + b)/2$;
2. The geometric mean $m_2 = \sqrt{ab}$;
3. The harmonic mean $m_3 = \dfrac{2ab}{a+b}$.

It is not hard to verify that

$$a \leq m_3 \leq m_2 \leq m_1 \leq b$$

with any equality holding if and only if $a = b$.

These means occur in a wide variety of real-life situations. The following examples and other applications can be found in [8] and [5, p. 39]. (1) If Bob's salary is $30,000 and Mike's is $40,000, then the average salary of the two is, of course, $35,000, the arithmetic mean of their salaries. (2) If an object is placed in the left-hand pan of a faulty beam balance, then it weighs 9 grams. Whereas if it is placed in the right-hand pan, then it weighs 4 grams. What is the true weight of the object? We leave this as an interesting exercise for the interested readers to verify that, the true weight of this object is 6 grams which is the geometric mean of 9 and 4, i.e., $6 = \sqrt{9 \times 4}$, instead of their arithmetic mean $6.5 = (9 + 4)/2$, as some readers may think. (3) Let us consider the average speed of a car in a round trip from the town A to the town B. Suppose that the average speed of the car from A to B is 40 mph, and the average speed on its way back from B to A is 60 mph. What is the average speed of the whole

journey? Many people would like to believe that it is 50 mph, the arithmetic mean of 40 and 60. Recall that the average speed of the car in the whole journey is the total distance traveled divided by the total time spent. Again, we leave it to the interested reader to verify that the correct answer is 48 mph, the harmonic mean of 40 and 60. As a matter of fact, when we deal with "average" value of some quantities involving ratios, harmonic mean comes into play very naturally. Stanley's article [15] "Fair means or foul?" and Machale's article [8] "What does 'mean' mean?", as well as the book [5], contain many examples of this sort and explain the significance of the study of various means. In general, for an n-tuple of positive real numbers $a = (a_1, a_2, \cdots, a_n) \in R_+^n$, the arithmetic mean, geometric mean, and harmonic mean are defined respectively as follows:

$$A_n(a) = \frac{1}{n}\sum_{i=1}^{n} a_i, \qquad \text{arithmetic mean,}$$

$$G_n(a) = \left[\prod_{i=1}^{n} a_i\right]^{1/n}, \qquad \text{geometric mean,} \quad \text{and}$$

$$H_n(a) = \frac{n}{\sum_{i=1}^{n} 1/a_i}, \qquad \text{harmonic mean.}$$

One of the most important classical inequalities for these means is, perhaps, the following:

$$A_n(a) \geq G_n(a) \geq H_n(a), \tag{1}$$

with any equality holding if and only if $a_1 = a_2 = \cdots = a_n$. Means and their inequalities, as well as different proofs and interesting applications in different fields, have always been among the favorite topics in many undergraduate research articles including [7, 8, 14, 15]. There are also many excellent books on means and related analytic and geometric inequalities such as [1, 5, 6, 10, 11]. One can find further relevant articles and discussions on means and their applications from these references. In this note, we shall take a closer look at the continuous mean discussed by Russell in [14], and present geometric interpretations for some particular cases of this continuous mean.

Continuous Mean. A *continuous mean*, or a *connected mean*, is a function that connects various means together as particular function values, or a family of means generated by various values of a single parameter. There has been a lot of work in the literature devoted to this subject. Some of the approaches are quite elementary but very appealing, see [7, 14], and others sophisticated [1, 10]. The following approach can be found in Russell's article [14]. For $a = (a_1, a_2, \cdots, a_n) \in \mathbf{R}_+^n$, let us consider the function

$$F(x)(a) = \left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{\frac{1}{x}}.$$

It is clear that $F(1)(a) = A_n(a)$, and $F(-1)(a) = H_n(a)$. By applying L'Hospital's rule we see that, $\lim_{x\to 0} F(x)(a) = G_n(a)$. Therefore, if we define $F(0)(a) = G_n(a)$, then $F(x)(a)$ is a continuous function defined everywhere and monotonically increasing. For special values of x, $F(x)(a)$ provides the basic means. Furthermore, if we denote by $\overline{a}$ and $\underline{a}$ the largest element and the smallest element of $a_1, a_2, \cdots, a_n$, respectively, then $\lim_{x\to+\infty} F(x)(a) = \overline{a}$, and $\lim_{x\to-\infty} F(x)(a) = \underline{a}$. For a fixed positive n-tuple a, one can easily graph $F(x)(a)$ using a computer or a graphing calculator. A natural question is "does $F(x_0)(a)$ have any particular significance for x_0 other than -1, 0, and 1?" The answer is affirmative. For instance,

$$F(2)(a) = \left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}\right)^{1/2}$$

is known as the *root mean* which is an important concept in analysis and statistics. Another important example is $F(1/3)(a)$. This is called the *Lorentz mean,* and is useful in the theory of equation of state for gases [7]. In economics, many indices are computed using geometric means. In statistics, many of the parametric statistical modeling techniques require the assumption of homogeneity of variance, additivity of structure, and normality of errors. Frequently in practice, all of these assumptions are not satisfied on the original scale of measurements of response. However, there may exist a transformation of the response variable on which these assumptions may be satisfied. Some of these transformations include the power transformation, $F(x)(a)$. The power transformations are commonly used in statistics and many other fields involving statistical inference. For a detailed explanation, refer to the references [5, p.30] and [9, p. 310]. Due to the limited space here, in the rest of this note, we shall

be concerned with only some geometric interpretations of $F(1/m)(a)$ for m = 2, 3.

Geometric Interpretations. We begin with the following question:

Let D_1 and D_2 be two circular disks with radii r_1 and r_2 respectively, and $r = \frac{r_1 + r_2}{2}$, the arithmetic mean of r_1 and r_2. Let D be a circular disk with radius r. It is not hard to see that

$$\text{Area}(D) \neq \frac{\text{Area}(D_1) + \text{Area}(D_2)}{2}$$

unless $r_1 = r_2$. However, since r is the average value of the two given radii, the area of D is certainly bigger than the area of the smaller disk and smaller than the area of the bigger disk. Which "mean" value of Area(D_1) and Area(D_2) should be used to evaluate Area(D)?

The above question can be answered in general as follows.

Theorem 1. Let $D_1, D_2, \cdots, D_n$ be n circular disks with radii $r_1, r_2, \cdots, r_n$ respectively, and let D be a circular disk with radius r, where $r = \frac{1}{n}\sum_{i=1}^{n} r_i$. Let $a = (a_1, a_2, \cdots, a_n)$ where $a_i = \pi r_i^2$, is the area of the ith circular disk, $i = 1, 2, \cdots, n$. Then

$$\text{Area}(D) = F\left(\frac{1}{2}\right)(a) \le A_n(a), \tag{2}$$

with equality holding if and only if $r_1 = r_2 = \cdots = r_n$, i.e., all disks are congruent.

Proof.

$$\text{Area}(D) = \pi r^2 = \pi\left[\frac{1}{n}\sum_{i=1}^{n} r_i\right]^2 = \left[\frac{\sqrt{\pi r_1^2} + \cdots + \sqrt{\pi r_n^2}}{n}\right]^2 = F\left(\frac{1}{2}\right)(a),$$

and then Theorem 1 follows from the inequality for the continuous mean

$$F\left(\frac{1}{2}\right)(a) \le F(1)(a) = A_n(a). \qquad \blacksquare \qquad (3)$$

Remark. A similar result holds for a family of squares and their average area, a family of equilateral triangles and their average area, as well as other families of regular polygonal curves and their average areas.

Likewise, if we apply the continuous mean inequality $F(1/3)(a) \le F(1)(a)$, then the following result follows immediately.

Theorem 2. Let $D_1, D_2, \cdots, D_n$ be n spheres with radii $r_1, r_2, \cdots, r_n$ respectively. Let D be a sphere of radius r, where $r = \frac{1}{n}\sum_{i=1}^{n} r_i$. Let $a = (a_1, a_2, \cdots, a_n)$ where $a_i = \frac{4\pi}{3} r_i^3$ is the volume of the ith sphere, $i = 1, 2, \cdots, n$. Then

$$\mathrm{Vol}(D) = F\left(\frac{1}{3}\right)(a) \le \frac{1}{n}\sum_{i=1}^{n} \mathrm{Vol}(D_i) = A_n(a),$$

with equality holding if and only if $r_1 = r_2 = \cdots r_n$, i.e., all spheres are congruent.

Theorem 2 holds, of course, for a family of cubes and their average volume. The reason we can compare an average value of areas (volumes) for a family of circular disks (spheres), or a family of squares (cubes) to a particular one with "average size" is that their areas (volumes) are simply power functions of a geometric invariant. For a circular disk and a sphere, this invariant is the radius, and for a square and a cube, it is the side length. Besides, we consider only a family of similar figures. A more general, but natural question is, for a given family of figures with different shapes, can we define a new figure as the one with an "average size", and how do we compare the average value of the areas (volumes) of those figures in the family with the area (volume) of the one with "average size"? Inspired by a number of interesting articles about geometric optimization problems in MAA journals, and other undergraduate mathematics journals such as [2, 3, 4, 17], we are able to deal with this problem in the next section.

Isoperimetric Quotient. Let C be a simple closed plane curve with perimeter L, and enclosing a domain of area A. The classical isoperimetric inequality asserts that

$$L^2 - 4\pi A \ge 0, \qquad (4)$$

with equality holding if and only if C is a circle. This is perhaps one of the oldest geometric inequalities which has been proved and reproved by many different methods and generalized in many different directions. Two immediate consequences of this inequality are:

(i) Of all plane figures of equal perimeter, the circle has the maximum area;

(ii) Of all plane figures of equal area, the circle has the minimum perimeter.

In geometry, the quantity $(4\pi A)/L^2$ is called the *isoperimetric quotient* of the curve C; it measures the deviation of C from circularity ([2, 12, 17]). It is interesting to note that the isoperimetric quotient is a similarity invariant of C. It depends only on the shape of the curve and is independent of its size. For instance, $(4\pi A)/L^2 = 1$ if C is a circle (no matter how large the radius r is), $(4\pi A)/L^2 = \pi/4$ if C is a square. In general, by a direct calculation, we can verify that if C is a regular n-sided plane polygon, then $(4\pi/A)/L^2 = \pi/\left(n \tan\frac{\pi}{n}\right)$. In [13, p.180], Pólya abbreviated this quotient as "I.Q." and thereby restated the *isoperimetric inequality* (4) as "Of all simple closed plane figures, the circle has the highest I.Q., 1". We shall denote by $I(C)$, the I.Q. of the curve C, and list a few examples for some common plane curves in the table 1.

C	circle	square	quadrant	rectangle 3:2	semicircle	sextant
$I(C)$	1	0.7854	0.774	0.75398	0.74668	0.7086

Table **1.**

Similarly, let D be a simple closed surface in $\mathbf{E}^3$ with volume V and surface area S. The I.Q. of D can be defined as

$$\frac{36\pi V^2}{S^3}$$

which is also a similarity invariant. The information contained in the following table 2 suggests that "Of all simple closed surfaces in $\mathbf{E}^3$, the sphere has the highest I.Q., 1."

D	sphere	icosahedron	dodecahedron	octahedron	cube	tedrahedron
$I(D)$	1	0.8288	0.7547	0.6045	0.5236	0.3023

Table **2.**

Isoperimetric quotients, their inequalities, and related problems are important in both mathematics and physics. The literature in this subject is very rich. In particular, we recommend the interested reader to [11,12,13].

Now, for the particular concerns of our problem here, we shall make a slight modification to the definition of I.Q.'s, and introduce the so-called *pseudo-I. Q.* for a geometric figure in $\mathbf{E}^2$ or $\mathbf{E}^3$ so that we can define an average sized figure for each given family of figures. For a simple closed plane curve C with perimeter L, area A, and I.Q. $I(C)$, the pseudo-I.Q. d_c of C is defined as

$$d_c = L\left(\sqrt{\frac{I(C)}{4\pi}}\right).$$

It is clear that d_c is a geometric invariant of C and $A = d_c^2$. For instance, $d_c = \sqrt{\pi}r$ if C is a circle of radius r, and $d_c = a$ if C is a square with side length a. Since pseudo-I.Q. depends on the perimeter of a curve, it is no longer a similarity invariant. Two congruent plane curves must have the same pseudo-I.Q., but the inverse is not true. We leave it to the interested reader to check that two plane figures having the same pseudo-I.Q. are not necessarily congruent (hint: consider two n-sided polygons with a given set of side lengths, and inscribed in a circle). Nevertheless, we still can claim that "of all simple closed plane curves with given perimeter, the circle has the highest pseudo-I.Q.!"

As the counterpart for three dimensional figures, we define the pseudo-I.Q. for a simple closed surface D in $\mathbf{E}^3$ with surface area S, enclosing a domain of volume V as

$$d_D = \sqrt{S}\left(\frac{I(D)}{36\pi}\right)^{1/6},$$

where $I(D)$ is the ordinary I.Q. of D. It is clear that such d_D is a well-defined geometric invariant for D, and $V = d_D^3$. Moreover, we also claim: "of all simple closed surfaces in $\mathbf{E}^3$ with given surface area, the sphere has the highest pseudo-I.Q.!" As many readers may have thought by now, there are natural higher dimensional analogues for isoperimetric quotients and related isoperimetric inequalities. There are even more general analogues in space forms other than Euclidean spaces [12]. Let us come back to the interpretations of the continuous mean and generalizations of Theorems 1 and 2.

Theorem 3. Let $D_1, D_2, \cdots, D_n$ be n simple closed plane curves with pseudo-I.Q.'s $d_1, d_2, \cdots, d_n$ respectively, and D be a simple closed plane curve with pseudo-I Q. d, where $d = \frac{1}{n}\sum_{i=1}^n d_i$. Let $a = (d_1^2, d_2^2, \cdots, d_n^2)$. Then

$$\text{Area}(D) = F\left(\frac{1}{2}\right)(a) \le A_n(a) = \frac{1}{n}\sum_{i=1}^n \text{Area}(D_i),$$

with equality holding when all D_i's are congruent.

Proof. From the definition of the continuous mean, we have that

$\text{Area}(D) = d^2 = F\left(\frac{1}{2}\right)(a)$, $\text{Area}(D_i) = d_i^2, i = 1, 2, \cdots, n$, and $F(1)(a) = A_n(a) = \frac{1}{n}\sum_{i=1}^n \text{Area}(D_i)$, thus Theorem 3 is simply again a consequence of the continuous mean inequality

$$F\left(\frac{1}{2}\right)(a) \le F(1)(a),$$

with equality holding when $d_1 = d_2 = \cdots = d_n$.

Likewise, we have the same result for simple closed surfaces in $\mathbf{E}^3$.

Theorem 4. Let $D_1, D_2, \cdots, D_n$ be n simple closed surfaces $\mathbf{E}^3$ with pseudo-I.Q's $d_1, d_2, \cdots, d_n$ respectively, and D be a simple closed surface in $\mathbf{E}^3$ with pseudo-I.Q. d, where $d = \frac{1}{n}\sum_{i=1}^n d_i$. Let $a = (d_1^3, d_2^3, \cdots, d_n^3)$. Then

$$\text{Vol}(D) = F\left(\frac{1}{3}\right)(a) \le A_n(a) = \frac{1}{n}\sum_{i=1}^n \text{Vol}(D_i),$$

with equality holding when all $D_i's$ are congruent.

There is no doubt that function values of $F(x)$ at other points may likely possess some unexpected properties, for examples, do $F(\pi)$, $F(1/\pi)$, $F(e)$, and $F(1/e)$ have any particular meanings? Pólya has pointed out some natural connections between inequalities for means and geometric inequalities [13], and Niven has used many basic mean inequalities, and written a very popular book, *Maxima and Minima Without Calculus* [11]. Our interpretations of the continuous mean here is simply another attempt to demonstrate the kind of connections which we believe could be among the interesting topics for undergraduate research in mathematics and statistics.

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A Note on 'Analytical Formulas for $\sum_{i=1}^{n}\left\lceil \frac{i}{p}\right\rceil$ and $\sum_{i=1}^{n}\left\lfloor \frac{i}{p}\right\rfloor$'

Hans J.H. Tuenter
York University

Recently, Sivakumar et al. [1] gave a derivation for the sums $\sum_{i=1}^{n}\lceil i/p\rceil$ and $\sum_{i=1}^{n}\lfloor i/p\rfloor$, where n and p are arbitrary positive integers. We give a simpler derivation by linking it directly to the sum $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.

For ease of notation denote $\alpha = \lceil n/p\rceil$ and $\beta = n \bmod p$, so that $n = \alpha p + \beta$. Observe that the sum $\sum_{i=1}^{n} i$ can be written as

$$\sum_{i=1}^{n} i = \sum_{j=1}^{\alpha}\sum_{k=1}^{p}((j-1)p+k) + \sum_{k=1}^{\beta}(\alpha p+k). \tag{1}$$

Dividing each summand by p and rounding up directly gives

$$\sum_{i=1}^{n}\lceil i/p\rceil = \sum_{j=1}^{\alpha}\sum_{k=1}^{p} j + \sum_{k=1}^{\beta}(\alpha+1) = \frac{1}{2}\alpha p(\alpha+1) + \beta(\alpha+1) = \frac{1}{2}(n+\beta)(\alpha+1). \tag{2}$$

Now observe that

$$\sum_{i=1}^{n}(\lfloor i/p\rfloor + 1 - \lceil i/p\rceil) = \sum_{i=1}^{n} 1_{\{p|i\}} = \alpha, \tag{3}$$

which gives

$$\sum_{i=1}^{n}\lfloor i/p\rfloor = \sum_{i=1}^{n}\lceil i/p\rceil - n + \alpha = \cdots = \frac{1}{2}\alpha(n+\beta-p+2). \tag{4}$$

Substitution of $\beta = n - \alpha p$ and $\alpha = \lfloor n/p\rfloor$ yields the formulas as given in [1].

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What is a Proof?

Clayton W. Dodge
University of Maine

Introduction

Proof might be defined as "that which convinces one of the truth of a statement." Of course, some people are more easily convinced than others. Then, too, a person who has never studied higher mathematics might not be convinced of the truth of Fermat's last theorem. On the other hand, such a person might be completely convinced of the truth of Goldbach's conjecture after trying a few cases. So, our definition of *proof* needs to be sharpened. Perhaps we might say that *proof* is "that which convinces a properly prepared person of the truth of a statement." This definition opens the question as to just what is *proper preparation*, but in any specific situation perhaps a reasonable agreement can be made. In any case, I shall not pursue that question.

It is, rather, my goal to present some proofs that should be useful in classrooms. Whether one uses the term *proof* or a substitute such as *demonstration* or *argument*, etc., is immaterial. Although using the term *proof* in high school and college classes tends to raise the blood pressure of many students and cause them to think in terms of highly formal arguments that no one can understand, students will respond positively to the question, "Why is this statement true?" When an answer is adequate, the student can be told that he or she has just given a proof. In such an informal manner, the concept of proof and the laws of logic can be presented and understood and even enjoyed. (Just consider how much time and effort students will expend solving logical games such as Rubic's Cube.) Even young children can and do understand the concept of a logical argument.

A real proof is tailored to the student's sophistication

A story I have used with children as young as four or five years old illustrates their understanding of logic. I tell them of the child who went to a party and had a piece of chocolate cake, a piece of carrot cake, and a piece of yellow cake. So he or she had four pieces of cake. I hold up a finger for each

named cake, so I am holding up three fingers when I say "four pieces of cake." If the child hasn't counted, I repeat the story. When he or she realizes the inconsistency, then I say, "Oh yes, afterwards the child had a *stomach ache*," which, of course, sounds like *stomach cake*.

For a child who is older, I ask how old he or she is. When the child answers, say, 7, I reply, "When I was your age, I was 9." That is, I add 1 or 2 years to their age.

Invariably, children recognize the illogic of either story. Children are used to thinking logically and they accept simple logical arguments. For example, without actually stating so, a child understands that 1 more than 1 is 2, 1 more than 2 is 3, and so forth. So, to add 4 and 3, many children will hold 4 fingers on one hand and 3 on the other. They then count "4, 5, 6, 7," starting with the 4 fingers and counting the three fingers as "5, 6, 7." That is, since $3 = 1 + 1 + 1$, then $4 + 3 = 4 + (1 + 1 + 1) = (4 + 1) + 1 + 1 = (5 + 1) + 1 = 6 + 1 = 7$. They may not have put the names *commutative* and *associative* to the laws of arithmetic they are using, but they do understand on an informal level much more than we give them credit for.

So, when and how should we introduce proof? The answer is, right in sub-primary, when they first enter school. Rules (or patterns) in mathematics should be noted and emphasized. For instance, when adding 5 + 2, it does not matter whether you start with the 2 or with the 5; the answer will be the same either way. It is quicker, however, to start with the larger number. Counting "5, 6, 7" is easier than counting "2, 3, 4, 5, 6, 7." By bringing such ideas to the attention of the students, and especially by showing them how they can save effort, students begin looking for such patterns. They look for logical arguments.

As another example from arithmetic, consider the question: Is zero an even number? It is easy just to answer the question with a "Yes." It is far more instructive to say, "I don't know. Let's work it out. What numbers are even?" The students, with a bit of guidance, can come up with patterns that will answer the question. The even numbers are 2, 4, 6, 8, ... Show them on a number line. Ask if the pattern can be continued to the left. Ask what you call numbers that are not even. Ask what numbers are odd. How can you decide whether a number is odd or even? What happens if you divide an odd number by 2? What happens if you divide an even number by 2? What happens if you divide zero by 2? How does zero fit into the patterns? Does it fit into either the odd or the even category? The students, then, will answer their own question, gaining understanding of both mathematics and proof in the process.

No one knows all the answers

It is easy for students to pose questions that you have never previously considered. And it is embarrassing to have to admit you do not know the answer, especially to youngsters. Therefore, if you make it a habit to answer a question with "I don't know, but let's look into it," then you are covered when you really don't know. Furthermore, it gives you a chance to think about it and look into it yourself. Many times the students will find their own answers. Occasionally, they themselves find an answer and then thank you for it! I have found this technique works well.

Again, why cannot one divide by zero? Since adding zero "does nothing" to a number, should not division by zero do the same? Of course, there are several ways to answer the question, depending upon the children's understanding at the time. If they understand division by fractions, one can use illustrations of division by smaller and smaller fractions such as 1/2, 1/10, 1/100, etc. If not, then a different argument must be used. Perhaps you remind the students that 6 ÷ 2 is the number of objects that would appear in each pile if 6 objects were divided into two equal piles. Certainly, 6 ÷ 1 makes sense as putting all 6 objects into one pile and is 6, but how can you arrange 6 objects in zero equal piles? It makes no sense.

Perhaps you recall a proof of the theorem that the midpoint of the hypotenuse of a right triangle is equidistant from all three vertices of the triangle. The usual proof involves dropping a perpendicular from that midpoint to one side of the triangle and then obtaining congruent right triangles. A much simpler and more visual proof follows. Visualize it or draw it as you read it. Let M be the midpoint of the hypothenuse of right triangle ABC with right angle at B. Cover the figure with tracing paper and put in a thumb tack at M. Trace the triangle and call the traced figure $A'B'C'$. Now rotate the traced figure a half turn about the thumb tack so that A' is at C and C' is at A. The two triangles now form a rectangle $ABCB'$. Because a rectangle has symmetry about its center M and its diagonals are equal, then MA, MB, MC, and MB' are all equal. In particular, we have proved the given theorem, that MA, MB, and MC are all equal, so we can tear off the tracing paper.

Any time you can make a picture, you provide powerful visual reinforcement

of the underlying concept.

Older students can examine the concept of symmetry in terms of translations, rotations, and reflections, but younger students can be satisfied with more elementary discussions of the meaning of symmetry. Although two-column proofs can be instructive, proofs in higher mathematics are given in the more informal paragraph form, which appears less formidable to the reader.

As a final example, consider this basic theorem: The base angles of an isosceles triangle are congruent (equal in size). There are many ways to prove this result, such as drawing from the apex of the triangle either the angle bisector, the altitude, or the median to the opposite side and then proving congruent the two triangles thus formed. Call the isosceles triangle ABC with apex at B. An interesting proof is to argue that triangles ABC and CBA are congruent by SAS, so angles A and C are congruent because they are corresponding parts of congruent triangles. An elementary variation of this proof nicely answers the question of why these base angles are congruent. Because sides AB and BC are congruent, then the triangle is symmetric in the bisector of angle B. If the triangle is flipped over (reflected in) that line, then its new position coincides with its old position. So its base angles are the same size. They are congruent.

Conclusion

A proof should be given or constructed when the student is ready for it, and the level of the proof should be consistent with the immediate needs of the student. If something is completely obvious, a proof is probably a waste of time. When there is a question or when something is not obvious, then a proof can be helpful. Some sort of proof should be given at such a time, or at least an explanation that a proof has to be postponed for some reason. The best proof is the proof that a student himself or herself constructs, perhaps with your help. Most importantly, a proof is a discussion or argument that convinces one that a given statement is true.

Scheduling Round-Robin Tournaments

James Oehmann (student)
University of North Florida

To schedule round-robin tournaments means to arrange N different teams, so that each team plays every other team exactly once. There is a method developed by Freud [1,2] which adds a "dummy team" to N if N is odd, so we may suppose that N is even in the following statement:

If you have N different teams, label the N teams with the integers 1,2,3,..., N-1, N. We have team I, with $I = N$, play team j, with $j = N$ and $j = I$, in the kth round if $I+j \equiv k \pmod{N-1}$. This schedules games for all teams in round k, except for team N and the one team I for which $2i \equiv k \pmod{N-1}$. There is one such team because the congruence $2x \equiv k \pmod{N-1}$ has exactly one solution with $1 \leq x \leq N-1$, since $(2, N-1) = 1$. We match this team I with team N in the kth round.

I have found an alternative method which is much simpler and does not use congruences. This method works for three or more teams and it will give you the same result as using the formula but it requires no calculations or understanding of higher mathematics. It only requires filling numbers into blanks by the following easy rules:

1.Give each team a number starting with 1. Make a chart with each team number on the top (start with 1 at the left and increase your numbers as you go to the right) and the tournament round numbers down the left side of your chart (start with 1 at the top and increase your numbers as you go down)

2.For an odd number of teams, write the highest team number along to main diagonal. Write the next highest along the diagonal above that. Keep decreasing the numbers as you fill in the diagonals. Below the main diagonal, fill the next diagonal with 1 and continue increasing the numbers until all diagonals are filled. [NOTE: You will need the same number of rounds as teams playing]

3. Now go back and any place that the number you wrote matches the team number, write "Bye" instead and that team will not have a game that round. Now, your chart is complete and your tournament schedule is complete. Table I is the example for five teams.

Team Number

	1	2	3	4	5
I	5	4	(3) Bye	2	1
II	(1) Bye	5	4	3	2
III	2	1	5	(4) Bye	3
IV	3	(2) Bye	1	5	4
V	4	3	2	1	(5) Bye

Table I

4. If there is an even number of teams, make a chart for the odd number preceding your number of teams and add a column for the last team. Anyone who has a "Bye" should be matched up with the new team that was just added so that everyone plays in every round. Write the new team number in place of the "Bye" and the corresponding number in the new team's column.

Table II is the example for six teams.

Team Number

	1	2	3	4	5	6
I	5	4	(3) 6	2	1	3
II	(1) 6	5	4	3	2	1
III	2	1	5	(4) 6	3	4
IV	3	(2) 6	1	5	4	2
V	4	3	2	1	(5) 6	5

Table II

Special Thanks to Dr. Jingcheng Tong for his help on this project.

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Math his passion, teaching his life...

Donald E. Miller
Laura Meyers (student)
Saint Mary's College

At its 1998 induction ceremony the Indiana Epsilon chapter posthumously inducted Milko Jeglic, a former professor and mentor at Saint Mary's College. We are pleased to share his story in hope that it might be an inspiration to aspiring mathematicians.

With Math as his passion, Milko Jeglic humbly endured hardship to make teaching his life. Educated and successful, Jeglic started his professional career teaching mathematics and pedagogy at a teacher's institute in Ljubljana, Slovenia. A communist invasion found Jeglic moving his family to Austria, only to bounce between refugee camps for over four years. Making the best of an adverse situation, Jeglic organized schools within the camps. Many would have accepted defeat after similar trials, but Jeglic's life started anew in the United States of America. Freedom was found, but professional respect was unrecognized at first. In the Saint Mary's community, the admiration for Jeglic was discovered again; he became known to his students as a classmate and to his colleagues as a mentor and role-model.

It was with pride that the Saint Mary's Epsilon chapter announced the posthumous induction of Milko Jeglic, a revered professor and gentleman of mathematics. Born June 2, 1901 as the sixth of eleven children in Rakek, Slovenia, Jeglic was an avid gymnast in his youth. For university studies, he headed to the University of Zagreb, Croatia where he studied Mathematics and Pedagogy, earning his professor's diploma in 1927. In his first professional position at the Teacher's Institute in Ljubljana, his talent and enthusiasm for teaching and education was recognized, resulting in his appointment as superintendent of the primary schools of Slovenia in only nine years. During this period as superintendent, he published six mathematics textbooks as well as a book on gymnastics.

At the end of World War II (1945), the communists invaded Slovenia. Jeglic took this as his cue to leave his native land and fled, with his wife and two sons, to neighboring Austria. At that time, Austria was under the British protectorate and conditions for refugees were less than ideal. It was the refugee camps of Austria that the Jeglic family called home for the next four and a half years.

To occupy his time, Jeglic stepped up to the challenge of organizing schools within the camps. To his credit these schools , unlike others of refugee encampments, were accredited by the British and Austrian authorities. Jeglic utilized his acute teaching abilities by teaching mathematics in truly multi-cultural classrooms on the secondary level. In no time at all, Jeglic became the principal of the school. Of his students, many went on to accomplish greatness; the recently named Cardinal Aloysius Ambrozic of Toronto is included in this elite list.

When it became evident that life for the Jeglic family was not going to improve in Austria, he again ventured to move his family. With the allure of freedom and the promise of a new start overshadowing the evident English language barrier, the family emigrated to the United States of America in 1949. Initially settling in the Mesabi Range of Minnesota (where he had a sponsor), and with no knowledge of the English language, Jeglic worked in the iron mines. The family spent but one winter there before moving to northern Illinois where Jeglic was reintroduced to campus life as a janitor at Lake Forest College.

Mathematics, whether the time period was the 1950's or is the 1990's, has always been a stress point for students; a person possessing knowledge of this academic area cannot stay hidden for long. Jeglic was no different — when students heard about this janitor who could do math, his tutelage was quickly in demand. The desire to return to the classroom intensified with each student he helped, for Jeglic was back in his element.

Through a childhood priest/friend, Jeglic became aware of an open faculty position at Saint Mary's College in Notre Dame, Indiana. After interviewing in the spring of 1952, he was overjoyed at the offer of a position as instructor for 1952-53 academic year. He was ecstatic to be back in the classroom. Colleagues related his first commencement in May of 1953, "Milko, in his academic splendor, sauntered down the lines of assembled faculty with tears streaming down his happy face saying, 'Today is the greatest day of my life. I am restored to my academic profession.'"

When Jeglic started his employment at Saint Mary's, a women's college, he made history by being the first male member of the mathematics department. This was a time when no mathematics was required for graduation. None-the-less, Jeglic, a quintessential European professor with a Slovenian accent, moustache, dapper dark suit and tie, made mathematics an enjoyable elective for women in the fifties. In his first years at Saint Mary's College, he engaged learners by calling them his classmates — they were learning mathematics, he was learning English. He brought compassion, love and understanding to a foreign world — the world of mathematics. With the influence of his foresight, by the time Jeglic left Saint Mary's College, mathematics had become a requirement for all graduates; the school was committed to developing the whole woman.

As part of the mathematics faculty in Notre Dame, Jeglic taught full-time at Saint Mary's College and chaired a Calculus course at the University of Notre Dame. In spite of this ambitious effort, he always maintained the highest of standards for both himself and his students. The New Math of the late fifties required a more abstract approach than he had known in Europe, but he adapted with ease. He welcomed new texts, new ideas, new course content, and spoke from experience when telling his students, "To learn mathematics you must suffer — not too much — but it will hurt a little." Though he taught a variety of classes, number theory was always his favorite.

His endearing concern and generosity can be seen through the *Professor Milko Jeglic Award for Achievement in Mathematics*, an annual monetary prize given to the student that has the highest number of quality points in Saint Mary's mathematics course through seven semesters. Jeglic, desirous of encouraging students to take more mathematics than the minimum, set the criteria and endowed the award.

To honor Milko Jeglic for his lifetime of scholarly service to mathematics, his students and colleagues, induction into Pi Mu Epsilon is timely at the end of his marvelous mathematical life. He will live in the hearts and memories of his students and colleagues. Now his name will be honored, as it should be, by a professional organization.

Acknowledgment. The authors with to thank Frank Jeglic, and Miriam Cooney, CSC, for their assistance with the historic information contained within this article.

BIOGRAPHICAL SKETCHES OF THE AUTHORS

Loi Nguyen is the vice president of the Vietnamese community at Jacksonville. As a father of two grown children, he is still a student at UNF seeking a bachelor's degree in mathematics. **Tu Tran** lived in Vietnam until 1991. He is a senior with a major in chemistry and a minor in computer science. He likes to create web sites. Tu will be a graduate student this summer. He loves his parents very much.

Paul Fishback received his undergraduate degree from Hamilton College and his doctorate from the University of Wisconsin. His mathematical interests are in analysis and dynamical systems. Since coming to Grand Valley, he has served as the faculty advisor for the Mathematics and Statistics Club. **Nicholas Ceglarek** graduated from Grand Valley State University in 1996, where in addition to studying mathematics, he set school records as a quarterback. He now teaches at Rockford High School in Rockford, Michigan. In addition to teaching, he coaches his high school football and mathematics team. **Tobias Moleski** graduated from Grand Valley State University in 1996, with degrees in physics and mathematics. He is now in the physics doctoral program at Oregon State University.

Prem N. Bajaj is an associate professor of Mathematics and Statistics at Wichita State University. He joined this school in 1968. He activated the Kansas Gamma Chapter of **Pi Mu Epsilon** in 1982 (and was its advisor until 1994). His hobbies are swimming and racquet ball. His area of research is Ordinary Differential Equations.

Paul S. Bruckman was born in Florence, Italy, became a naturalized U.S. citizen in 1950, and received his M.S. degree in mathematics from the University of Illinois at Chicago in 1974. From 1960 through 1990, he was employed with private actuarial firms, most recently as a pension plan actuary. Mr. Bruckman has been and continues to be a frequent contributor to **The Fibonacci Quarterly** and the **Pi Mu Epsilon Journal**. His primary mathematical interests are rooted in number theory, linear recurrences and pseduoprimes.

Masakazu Nihei is a teacher of mathematics at Fujishiro High School in Japan. He received his M. S. in 1975 from Rikkyo University. His research interests are graph theory and mathematical education.

Subhash C. Saxena received his Ph.D. in Differential Geometry from the University of Delhi, India. For the last 35 years, he has taught at several universities in the United States. Dr. Saxena has published in the areas of geometry, analysis, algebra and discrete mathematics. He has authored a book entitled "Introduction to Real Variable Theory." He has won the Distinguished Teaching Award at his campus and the Amoco Outstanding Teaching Award for all 4-year campuses of the University of South Carolina system. His biographical sketch is listed in may Who's Who publications. Recently, he has been included in the current (15^{th}) edition of 'Who's Who in the World'.

Arvind Shah is a professor of statistics and **Xin-Min Zhang** is an associate professor of mathematics in the Department of Mathematics and Statistics at the University of South Alabama.

Hans J. H. Tuenter received the B. Sc. degree in Applied Mathematics, and the M. Sc. degree in Operational Research & Stochastics from Twente University of Technology, Enschede, The Netherlands, and the Ph. D. degree in Industrial Mathematics from the University of Birmingham, England. He is currently a post doctoral fellow in Management Science at the Schulich School of Business, York University, Toronto, Canada.

Clayton W. Dodge received his B.A. and M.A. degrees from the University of Maine in 1956 and 1959. He taught there for 41 years, retiring as Professor Emeritus of Mathematics in 1997. His interests include geometry, teacher education, and problems. He was an editor for the Elementary Problem Department of the *American Mathematical Monthly* and has served as Problems Editor for this journal since 1980.

James Oehmann is a senior with a Mathematics major and a Physics minor. He will graduate in the spring of 1999. He has lived in Jacksonville since 1990 and is getting married in June.

Donald E. Miller is associate professor and Chair of the Department of

Mathematics at Saint Mary's College, Notre Dame. He is a member of the Education Committee of the Society for Industrial and Applied Mathematics and has served as Chair of the Indiana Section of the Mathematical Association of America. His consulting and research activities focus on modeling and applied statistics; current research with a colleague in Political Science involves statistical analysis of the politics related to state approved gambling. Having been recruited by Jeglic, they served as colleagues for more than a decade. **Laura Meyers** is a 1998 graduate of Saint Mary's College and a member of **Pi Mu Epsilon**. She holds a Bachelor of Arts degree in Mathematics with a minor in Business Administration. Highlights of her Saint Mary's career include completing a year long senior comprehensive research project, with Donald Miller as her advisor, that focused on Markov Chains and spending a semester abroad in Fremantle, Australia, during her junior year. She is currently employed by Ernst & Young LLP as a Management Consultant in Chicago.

MISCELLANY

Erratum

The last paragraph on page 684 (Volume 10, Number 8) should have read:

> The **American Mathematical Society** has given **Pi Mu Epsilon** a grant to be used as monetary awards for excellent student presentations. The six speakers indicated above each received $100. **The National Security Agency** has provided money to pay for student subsistence.

A Mathematical Christmas Tree with Ornaments

In the carol "The Twelve Days of Christmas," it is known that the total number of gifts given over the course of the twelve days is 364. Rotation the standard multiplication table produces the following "Christmas tree".

1
2 2
3 4 3
4 6 6 4
5 8 9 8 5
6 10 12 12 10 6
7 12 15 16 15 12 7
8 14 18 20 20 18 14 8
9 16 21 24 25 24 21 16 9
10 18 24 28 30 30 28 24 18 10
11 20 27 32 35 36 35 32 27 20 11
12 22 30 36 40 42 42 40 36 30 22 12

Notice that by summing the entries in the twelfth row of this Christmas tree one is finding the total number of gifts received during the twelve days of Christmas.

In general, summing the elements in the nth row of the above tree gives the following.

$$S(n) = n(1) + (n - 1)2 + \cdots + 2(n - 1) + (1)n$$

$$= \sum_{i=1}^{n} (n - i + 1)i$$

$$= \sum_{i=1}^{n} [(n + 1)i - i^2]$$

$$= (n + 1) \sum_{i=1}^{n} 1 - \sum_{i=1}^{n} i^2$$

$$= (n + 1)n(\ + 1)/2 - n(n + 1)(2n + 1)/6$$

$$= n(n + 1)\ [3(n + 1) - (2n + 1)]/6$$

$$= n(n + 1)(n + 2)/6$$

This formula gives the number of gifts accumulated at any day n. In particular, $S(12) = (12)(13)(14)/6 = 364$ and $S(365) = (365)(366)(367)/6 = 8171255$.

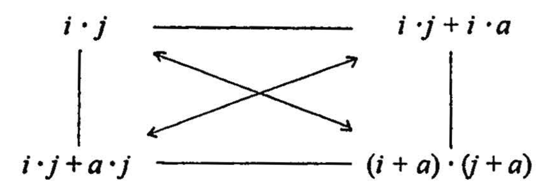
It should be noted that $S(n) = \binom{n+2}{3}$.

The Christmas tree is bedecked with 'diamond shaped ornaments' formed from multiplicands. A typical ornament is shown below.

```
            15
       20         18
  25        24         21
       30         28
            35
```

In this ornament, notice that $15 \cdot 35 = 25 \cdot 21$. To see that a similar property holds in general, rotate the Christmas tree back to the standard multiplication

table. Now, the diamond pattern mentioned can be translated into a square pattern. Choose any element in the multiplication table--say the number in the ith row and jth column. The value of this number is $i \cdot j$. In forming the square pattern with "integral side length a," one would have the following four numbers in the table.



Now, checking the "ornament pattern" becomes a matter of verifying that

$$i \cdot j \cdot (i + a) \cdot (j + a) = (i \cdot j + i \cdot a) \cdot (i \cdot j + a \cdot j).$$

This is a simple exercise left to the reader.

PROBLEM DEPARTMENT

Edited by Clayton W. Dodge
University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. E-mail: dodge@gauss.umemat.maine.edu. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed to arrive by July 1, 1999. Solutions by students are given preference.

Problems for Solution

940. *Proposed by Mike Pinter, Belmont University, Nashville, Tennessee.*

In the following base ten alphametic determine the maximum value for *MONEY*:

$$DAD + SEND = MONEY.$$

941. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, Pennsylvania.*

Let $a_1 = 1$, $a_2 = k > 2$, and for $n > 2$, $a_n = ka_{n-1} - a_{n-2}$.

a) Show that the general term a_n is given by

$$a_n = \frac{B^n - B^{-n}}{B - B^{-1}}, \text{ where } B = \frac{k + \sqrt{k^2 - 4}}{2}.$$

*b) Find a suitable expression for the sum S_n of the first n terms.

942. *Proposed by John S. Spracker, Western Kentucky University, Bowling Green, Kentucky.*

Calculate the following "sums" of the form $\Sigma_{n=1}^{\infty} a_n$ for each given sequence $\{a_n\}$ and given addition $\oplus$. To deal with nonassociative operations define $S_1 = a_1$ and $S_{n+1} = S_n \oplus a_n$ for $n > 0$.

a) On R^+ let $a \oplus b = 1/a + 1/b$ and take $a_n = n$. Then $S_1 = 1$, $S_2 = 1 + 1/2 = 3/2$, $S_3 = 2/3 + 1/3 = 1$, ...

b) On R^+ let $a \oplus b = ab/(a + b)$ and take $a_n = 1/n$.

c) On R^+ let $a \oplus b = \sqrt{ab}$ and take $a_n = 1/n$.

d) On R let $a \oplus b = \cos(a + b)$ and take $a_n = 2\pi n$.

943. *Proposed by Paul S. Bruckman, Edmonds, Washington.*

Let $\alpha = (1 + \sqrt{5})/2$ and let F_n denote the nth Fibonacci number, so that $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for $n > 0$. For $n = 1, 2, \ldots$ define

$$U_n = \alpha^n \prod_{k=1}^{n} \frac{F_{2k}}{F_{2k+1}} \text{ and } V_n = \alpha^n \prod_{k=1}^{n} \frac{F_{2k-1}}{F_{2k}}.$$

Prove that $U = \lim_{n\to\infty} U_n$ and $V = \lim_{n\to\infty} V_n$ exist. If possible, evaluate U and V in closed form.

944. *Proposed by David Iny, Baltimore, Maryland.*

Evaluate

$$\int_0^{\infty} \frac{dx}{x + e^x}.$$

***945.** *Proposed by the late Jack Garfunkel, Flushing, New York.*

Let A, B, C be the angles of a triangle and A', B', C' those of another triangle with $A \ge B \ge C$, $A > C$, $A' \ge B' \ge C'$, and $A' > C'$. Prove or disprove that

$$\text{if } A - C \ge 3(A' - C'), \text{ then } \sum \cos\frac{A}{2} \le \sum \sin A'.$$

946. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, Pennsylvania.*

Let M be a point inside (outside) triangle ABC if $\angle A$ is acute (obtuse) and let $m\angle MBA + m\angle MCA = 90°$.

a) Prove that $(BC\cdot AM)^2 = (AB\cdot CM)^2 + (CA\cdot BM)^2$.

b) Show that the Pythagorean theorem is a special case of the formula of part (a).

947. *Proposed by Paul S. Bruckman, Edmonds, Washington.*

In the card game of hearts, a regular deck of 52 cards is dealt to four players. An assigned player leads off, and tricks are taken by rules that need not concern us here. Each heart-suit card is assigned a value of 1 point, and the queen of spades has a value of 13 points; thus, the total value of each hand is 26 points. Your score for any hand is the sum of the points in the tricks you have taken. If one player, however, takes *all* 26 points in any hand, then that player is awarded 0 points and each of the other players is burdened with 26 points.

The object of the game is to accumulate the fewest points. Hands continue to be played until at least one player has 100 or more points, at which time the player with the fewest points is declared the winner of the game. Ties are possible. Suppose the winner's total gain after a game is the total of the differences between his score and that of each other player. At \$1 a point, what is the winner's maximum possible total gain per game?

948. *Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.*
All six faces of a cube 4 inches on a side are painted red. Then the cube is chopped into 64 smaller 1-inch cubes. The "inside" faces are left unpainted. The 64 small cubes are put into a box and one is drawn at random, and tossed. Find the probability that when it comes to rest its upper face will be red.

949. *Proposed by Charles Ashbacher, Decisionmark, Cedar Rapids, Iowa.*

In a collection of problems edited by Dumitrescu and Seleacu [1] a positive integer is said to be a *Smarandache pseudo-odd(even) number* if some permutation of its digits is odd(even). For example, 12345678 is both Smarandache pseudo-even and pseudo-odd since 12456783 is odd. A positive integer is said to be a *Smarandache pseudo-multiple* of the positive integer k if some permutation of its digits is divisible by k.

a) Prove that if a positive integer is chosen at random, the probability that it is Smarandache pseudo-odd is 1.

b) Prove that if a positive integer is chosen at random, the probability that it is Smarandache pseudo-even is 1.

c) Prove that if a positive integer is chosen at random, the probability that it is a Smarandache pseudo-multiple of 3 is 1/3.

d) Prove that if a positive integer is chosen at random, the probability that it is a Smarandache pseudo-multiple of 5 is 1.

Reference

1. c. Dumitrescu and V. Seleacu, *Some Notions and Questions in Number Theory*, Erhus University Press, 1994.

950. *Proposed by S. B. Karmakar, Piscataway, New Jersey.*

Let $c > b > a > 0$ be the lengths of the sides of an obtuse triangle; let m be a prime and n an even positive integer such that $1 < d = m/n < 2$. Without using Fermat's last theorem prove that the equation

$$a^d + b^d = c^d$$

cannot be satisfied if a, b, and c are relatively prime in pairs.

951. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

An ant crawls along the surface of a *dicube*, a $1 \times 1 \times 2$ rectangular block.

a) If the ant starts at a corner, where is the point farthest from it? (It is not the opposite corner!)

b) Find two points that are farthest apart from each other on the surface of the dicube?

952. *Proposed by Peter A. Lindstrom, Batavia, New York.*

Let A, B, C denote the measures of the angles and a, b, c the lengths of the opposite sides of a triangle. Show that

$$\sin A \sin B + \sin B \sin C + \sin C \sin A = \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)(bc+ca+ab)}{4a^2b^2c^2}.$$

Solutions

910. [Spring 1997, Spring 1998] *Proposed by William Chau, New York, New York.*

A triangle whose sides have lengths a, b, and c has area 1. Find the line segment of minimum length that joins two sides and separates the interior of the triangle into two parts of area α and $1 - \alpha$, where α is a given number between 0 and 1.

II. *Comment by the Proposer.*

Since $\cos A = (b^2 + c^2 - a^2)/(2bc)$ and $1 = (1/2)bc\sin A$, then in Solution I we may write

$$\tan\frac{A}{2} = \frac{1 - \cos A}{\sin A} = \frac{a^2 - (b - c)^2}{4} = \frac{(a - b + c)(a + b - c)}{4}.$$

Therefore, $z = \sqrt{\alpha(a - b + c)(a + b - c)}$ and the minimum length of MN is

$$\min\left(\sqrt{\alpha(a-b+c)(a+b-c)}, \sqrt{\alpha(b-c+a)(b+c-a)}, \sqrt{\alpha(c-a+b)(c+a-b)}\right).$$

914. [Fall, 1997] *Proposed by Peter A. Lindstrom, Batavia, New York.*

Solve this base ten addition alphametic, dedicated to the memory of the late Leon Bankoff:

$$FRIEND + INDEED = BANKOFF.$$

Solution by Aaron Kerr, student, Alma College, Alma Michigan.

Clearly, $B = 1$, $I \neq 0$, $R = 9$ or 0, F is 2, 4, 6, or 8, $D \neq 0$ or 5, $E \neq 0$, $D \neq 9$ and $I \neq 9$ because otherwise $R = 9$, too, and $E \neq 9$ since otherwise $O = 9$, too. Suppose $R = 0$. Then there is no carry to the next column, so $F + I \geq 12$ and $I + D < 10$. Hence $F \neq 2$. If $F = 4$, then $D = 2$ or 7 and $I \geq 8$, making $I + D > 9$. So $F \neq 4$. If $F = 6$, then $D = 3$ or 8 and $I \geq 7$, so again $I + D > 9$. Hence $F \neq 6$. Only $F = 8$ is possible. Then $D = 4$ or 9 and $I \geq 4$. Now $D = 4$, so $I = 5$, $K = 9$, and $A = 3$. From the tens column N and E are 2 and 6. From the hundreds column $E \neq 6$, so $E = 2$ and $O = 4$, contradicting that $D = 4$. Therefore, $R \neq 0$, so $R = 9$.

From the hundred thousands column $A \leq 6$ since neither addend can be 9. Suppose $F = 2$. Then $I = 7$, $A = 0$, $D = 6$, N and E are 8 and 3, and $k = 4$. Then $E = 8$ and $O = 7$, contradicting that $I = 7$.

Suppose $F = 4$. Then $I = 5$, 7, or 8, and $D = 2$ or 7. If $I = 5$, then $A = 0$ and $D = 7$. Now N and E cannot be evaluated. If $I = 7$, then $D = 2$ and $A = 2$ also. So $I = 8$ and $A = 3$. Now $D = 2$ or 7, neither one of which leads to available values for N and E. Hence we cannot have $F = 4$.

We try $F = 6$. Then $D = 3$ or 8. If $D = 3$, then N and E are 2 and 4, so $O = 4$ or 8. Thus $N = 2$, $E = 4$, and $O = 8$. Now $I = 5$ or 7, and $A = 2$ or 4 respectively. So $D \neq 3$ and we set $D = 8$. Then N and E are 0 and 5, or 2 and 3. If $I = 3$, then $A = 0$, so $I \neq 3$. Similarly, if $I = 5$, then $A = 2$. So $I = 7$ and $A = 4$. Now $E \neq 0$, 2, or 3 since then $O = 0$, 4, or 6. Similarly $E \neq 5$ since then both N and O would be 0.

Therefore, F must be 8. With this information the rest of the values fall into line quickly. We must have $D = 4$ and $N + E = 8$. So N and E are 2 and 6, or 3 and 5. From the thousands column, $I \geq 5$. From the hundred thousands column, $I = 6$ and $A = 5$ conflicts with N and E, so $I = 7$ and $A = 6$. Now $N = 3$, $E = 5$, and $O = 0$. Finally, $K = 2$. Cool. It works. We have

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  897534
+ 734554
 1632088.
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For an interesting demonstration of alphametics, check out the website http://www.ceng.metu.edu.tr/~selcuk/alphametic/index/html. This page is an alphametic solver. Put in any puzzle and it will give you all possible solutions.

Also solved by Alma College Problem Solving Group, *MI*, Ben Andrews, *Hendrix College, Conway, AR*, Charles D. Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA*, Frank P. Battles, *Massachusetts Maritime Academy, Buzzards Bay*, Karen Bernard, *Arkansas Governor's School, Conway*, Scott H. Brown, *Auburn University, AL*, Paul S. Bruckman, *Edmonds, WA*, Genafer Cantrell, *Arkansas Governor's School, Conway*, Yun Choi, *Arkansas Governor's School, Conway*, Jonathan D. Croft, Wilson Davis, *Hendrix College, Conway, AR*, Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville*, Mark Evans, *Louisville, KY*, Stephen I. Gendler, *Clarion University of Pennsylvania*, Richard I. Hess, *Rancho Palos Verdes, CA*, Joe Howard, *New Mexico Highlands University, Las Vegas*, Tom Huynh, *Arkansas Governor's School, Conway*, Carl Libis, *University of Alabama, Tuscaloosa*, Mimi Liu, *Arkansas Governor's School, Conway*, Yoshinobu Murayoshi, *Okinawa, Japan*, Khai To, *Arkansas Governor's School, Conway, and the* Proposer.

***915.** [Fall, 1997] *Proposed by the late John Howell, Littlerock, California.*

Prove or disprove that, if $n \geq 0$, $k \geq 0$, and $n + k \geq 1$, then

$$n! = (n+k)^n - \binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \cdots + (-1)^n k^n.$$

I. *Solution by Cecil Rousseau, The University of Memphis, Memphis, Tennessee.*

The conditions on k are unnecessary. Let Δ denote the difference operator: $\Delta f(x) = f(x+1) - f(x)$. Then

$$\Delta^2 f(x) = [f(x+2) - f(x+1)] - [f(x+1) - f(x)] = f(x+2) - 2f(x+1) + f(x),$$

$$\Delta^3 f(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x),$$

and in general,

$$\Delta^n f(x) = \sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} f(x+r).$$

In view of the symmetric property $\binom{n}{r} = \binom{n}{n-r}$ the right hand side of the given identity is seen to be $\Delta^n f(x)$ where $f(x) = x^n$.

Clearly $\Delta x^n = (x+1)^n - x^n = nx^{n-1} + p_{n-2}(x)$, where $p_{n-2}(x)$ is some polynomial in x of degree $n-2$, $\Delta^2 x^n = n(n-1)x^{n-2} + p_{n-3}(x)$, and so on. It follows by induction that $\Delta^n x^n = n!$ for all x.

Also solved by Scott H. Brown, *Auburn University, AL*, Paul S. Bruckman, *Edmonds, WA*, Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville*, Richard I. Hess, *Rancho Palos Verdes, CA*, Murray S. Klamkin, *University of Alberta, Canada*, Carl Libis, *University of Alabama, Tuscaloosa*, Florian Luca, *Syracuse University, NY*, David E. Manes, *SUNY College at Oneonta*, Bob Prielipp, *University of Wisconsin-Oshkosh, and* H.-J. Seiffert, *Berlin, Germany*.

Comment by the Editor. The proposer remarked "I am not sure if this is 'well known' or not." Euler pointed out his article "Exponential Differences Revisited," *Journal of Recreational Mathematics*, 24(1992) pp. 123-124. Libis referred to his University of Alabama doctoral thesis "Generalizations of Bernoulli and Other Polynomials," August 1998. Luca cited a talk by Sebastián Martín Ruiz of University of Sevilla at the *First International Conference on Smarandache Notions in Number Theory* in Craiova, Romania, August 1997, based on his paper "An Algebraic Identity. Consequence: Wilson's Theorem," submitted to *The Mathematical Gazette*. Prielipp mentioned Problem B-5 in the 1976 Putnam Mathematical Competition, *The American Mathematical Monthly* 85(1978) pp. 29 and 32. Seiffert specified R. Wyss, "Lösung von Aufgabe 983," *Elem. Math.* 44(1989) pp. 48-49. It does appear that the formula is well known.

916. [Fall, 1997] *Proposed by Morris Katz, Macwahoc, Maine.*

Prove these two formulas:

$$1^2(2n-1)^2 + 2^2(2n-2)^2 + \cdots + n^2n^2 = \frac{1}{30}n(n+1)(16n^3 - n^2 + n - 1)$$

and

$$1^2(2n-1)^2 - 2^2(2n-2)^2 + \cdots + (-1)^{n+1}n^2n^2 = \frac{1}{2}n[1 - (-1)^n n^3].$$

Solution by Andrea Vujan, student, Wheaton College, Wheaton Illinois.

Let A denote the left side of the first formula and B that of the second one. Then we have

$$A = 1^2(2n-1)^2 + 2^2(2n-2)^2 + \cdots + n^2n^2$$

$$= \sum_{i=1}^{n} i^2(2n-i)^2 = 4n^2\sum_{i=1}^{n} i^2 - 4n\sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} i^4$$

$$= 4n^2\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) - 4n\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) + \left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right)$$

$$= \frac{16n^5}{30} + \frac{n^4}{2} - \frac{n}{30} = \frac{n}{30}(n+1)(16n^3 - n^2 + n - 1)$$

by using well known formulas for Σi^2, Σi^3, and Σi^4.

Next, let $C = A - B$. If n is even, we let $n = 2k$ and then

$$C = 2\sum_{i=1}^{n/2}(2i)^2(2n-2i)^2 = 2^5\sum_{i=1}^{k} i^2(2k-i)^2$$

$$= 2^5\left(\frac{16k^5}{30} + \frac{k^4}{2} - \frac{k}{30}\right) = \frac{16n^5}{30} + n^4 - \frac{16n}{30}.$$

So

$$B = A - C = -\frac{n^4}{2} + \frac{n}{2} = \frac{n}{2}[1 - (-1)^n n^3].$$

If n is odd, we let $n = 2k+1$ and then

$$C = 2\sum_{i=1}^{(n-1)/2}(2i)^2(2n-2i)^2 = 2^5\sum_{i=1}^{k} i^2(2k+1-i)^2$$

$$= 2^5\left(4k^2\sum_{i=1}^{k} i^2 + \sum_{i=1}^{k} i^2 + \sum_{i=1}^{k} i^4 + 4k\sum_{i=1}^{k} i^2 - 2\sum_{i=1}^{k} i^3 - 4k\sum_{i=1}^{k} i^3\right)$$

$$= 2^5\left(\frac{8k^5}{15} + \frac{4k^4}{3} + \frac{4k^3}{3} + \frac{2k^2}{3} + \frac{2k}{15}\right) = \frac{16n^5}{30} - \frac{16n}{30}.$$

Again

$$B = A - C = \frac{n^4}{2} + \frac{n}{2} = \frac{n}{2}[1 - (-1)^n n^3].$$

Also solved by Paul S. Bruckman, *Edmonds, WA*, Deborah Carrillo, *Wheaton College, IL*, William Chau, *A T & T Laboratories, Middletown, NJ*, Kenneth B. Davenport, *Pittsburgh, PA*, Russell Euler *and* Jawad Sadek (*three solutions*), *Northwest Missouri State University, Maryville*, Mark Evans, *Louisville, KY*, Richard I. Hess, *Rancho Palos Verdes, CA*, Joe Howard, *New Mexico Highlands University, Las Vegas*, Carl Libis, *University of Alabama, Tuscaloosa*, Peter A. Lindstrom, *Batavia, NY*, David E. Manes, *SUNY College at Oneonta*, Yoshinobu Murayoshi, *Okinawa, Japan*, Bob Prielipp, *University of Wisconsin-Oshkosh*, Monte J. Zerger, *Adams State College, Alamosa, CO*, *and the* Proposer.

917. [Fall, 1997] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Determine necessary and sufficient conditions on the real numbers w_1, w_2, ..., w_n so that for all vectors $\mathbf{v}_i$ in E^m,

$$|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n|^2 \le w_1|\mathbf{v}_1|^2 + w_2|\mathbf{v}_2|^2 + \cdots + w_n|\mathbf{v}_n|^2.$$

Solution by David Iny, Baltimore, Maryland.

First observe that a necessary condition is that each $w_i \ge 1$ by taking exactly one $\mathbf{v}_i$ to be a nonzero vector. By ignoring the trivial case in which $\Sigma_{i=1}^n \mathbf{v}_i = 0$, we note that the inequality holds if and only if we replace each vector $\mathbf{v}_i$ with $\mathbf{v}_i/|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n|$. That is, the given inequality holds if and only if the minimum value of

$$f(\mathbf{v}_1,\ldots,\mathbf{v}_n) = \sum_{i=1}^{n} w_i|\mathbf{v}_i|^2,$$

subject to the constraint $|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n| = 1$, is $f_{\min} \ge 1$. The problem reduces to finding $f_{\min}$ in terms of the w_i. The obvious idea is to minimize the Lagrangian

$$L(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \lambda) = \sum_{i=1}^{n} w_i(\mathbf{v}_i^T\mathbf{v}_i) + \lambda\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{v}_i^T\mathbf{v}_j - 1\right),$$

which is well posed when each $w_i \ge 1$. Taking partials, we find that

$$2w_i\mathbf{v}_i + \lambda\left(2\sum_{j=1}^{n}\mathbf{v}_j\right) = 0.$$

Thus

$$\mathbf{v}_i = \left(\frac{1}{w_i}\right)\mathbf{v} \quad \text{where} \quad \mathbf{v} = -\lambda\sum_{j=1}^{n}\mathbf{v}_j.$$

Then the given inequality holds if and only if

$$\left(\sum_{i=1}^{n}\frac{1}{w_i}\right)^2|\mathbf{v}|^2 \le \sum_{i=1}^{n}\left(\frac{1}{w_i}\right)|\mathbf{v}|^2.$$

That is, if and only if

$$\sum_{i=1}^{n} \frac{1}{w_i} \le 1.$$

Also solved by H.-J. Seiffert, *Berlin, Germany, and the* Proposer. *Four incorrect solutions were also received.*

918. [Fall, 1997] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Evaluate the integral

$$I = \int_0^{\pi/3} \ln(1 + \sqrt{3}\tan x)\, dx.$$

I. *Solution by N. R. Nandakumar, Delaware State University, Dover, Delaware.*

More generally, consider the integral

$$I(a) = \int_0^a \ln(1 + k \tan x)\, dx,$$

where $0 < a < \pi/2$ and $k = \tan a$. With the substitution $y = a - x$ the integral becomes

$$I(a) = -\int_a^0 \ln[1 + k\tan(a-y)]\, dy = \int_0^a \ln\left(1 + k\frac{\tan a - \tan y}{1 + \tan a \tan y}\right) dy$$

$$= \int_0^a \ln\left(\frac{1+k^2}{1 + k\tan y}\right) dy = \int_0^a \ln(1 + k^2)\, dy - I(a).$$

Thus we have

$$I(a) = \frac{1}{2}\int_0^a \ln(1 + k^2)\, dy = \frac{1}{2} a \ln(1 + k^2).$$

Then, since $k = \tan(\pi/3) = \sqrt{3}$, we have

$$I(\pi/3) = \int_0^{\pi/3} \ln(1 + \sqrt{3}\tan x)\, dx = \frac{\pi \ln 4}{6} = \frac{\pi \ln 2}{3}.$$

II. *Solution by Shiva K. Saksema, University of North Carolina at Wilmington, Wilmington, North Carolina.*

By letting $u = \pi/3 - x$ we obtain a result that we shall need shortly:

$$\int_0^{\pi/3} \ln[\cos(\pi/3 - x)]\, dx = \int_{\pi/3}^0 \ln(\cos u)(-du) = \int_0^{\pi/3} \ln(\cos u)\, du.$$

Now we have

$$I = \int_0^{\pi/3} \ln\left(\frac{(1/2)\cos x + (\sqrt{3}/2)\sin x}{(1/2)\cos x}\right) dx = \int_0^{\pi/3} \ln\left(\frac{\cos(\pi/3 - x)}{(1/2)\cos x}\right) dx$$

$$= \int_0^{\pi/3} \ln[\cos(\pi/3 - x)]\, dx - \int_0^{\pi/3} \ln(\cos x)\, dx + \int_0^{\pi/3} \ln 2\, dx$$

$$= \int_0^{\pi/3} \ln 2\, dx = \frac{\pi \ln 2}{3}.$$

III. *Solution by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico.*

By *Math. Gazette*, Note 70.24, vol. 70 (452) (1986) p.143, the trapezoidal method gives the exact value. Using the partition $\{0, \pi/6, \pi/3\}$, we have

$$I = \frac{\pi}{6}\left[\frac{1}{2}\ln 1 + \ln\left(1 + \sqrt{3}\tan\frac{\pi}{6}\right) + \frac{1}{2}\ln\left(1 + \sqrt{3}\tan\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{6}\left(\ln 2 + \frac{1}{2}\ln 4\right) = \frac{\pi \ln 2}{3}.$$

Also solved by Frank P. Battles, *Massachusetts Maritime Academy, Buzzards Bay,* Paul S. Bruckman (*2 solutions*), *Edmonds, WA,* Charles R. Diminnie and Roger Zarnowski, *Angelo State University, San Angelo, TX,* Robert C. Gebhardt, *Hopatcong, NJ,* Richard I. Hess, *Rancho Palos*

Verdes, CA, David Iny, *Baltimore MD*, Peter A. Lindstrom, *Batavia, NY*, Cecil Rousseau, *The University of Memphis, TN*, H.-J. Seiffert, *Berlin, Germany*, George Tsapakidis, *Agrinio, Greece*, and *the* Proposer.

919. [Fall, 1997] *Proposed by the Editor.*

Erect directly similar nondegenerate triangles *DBC, ECA, FAB* on sides *BC, CA, AB* of triangle *ABC*. At *D, E, F* center circles of radii $k{\bullet}BC$, $k{\bullet}CA$, $k{\bullet}AB$ respectively for fixed positive k. Let P be the radical center of the three circles. If P lies on the Euler line of the triangle, show that it always falls on the same special point.

Solution by the Proposer.

The *power of a point P with respect to a given circle* is the product of the signed distances from that point to any two points on the circle collinear with P. If the circle has radius r and center O, then the power of P is equal to $OP^2 - r^2$. It is positive, zero, or negative according as P lies outside, on, or inside the circle. The *radical axis* of two given nonconcentric circles is the locus of all points P that have equal powers with respect to the two circles. It is a straight line perpendicular to their line of centers. If the circles intersect, it is their common chord. The *radical center* of three circles with noncollinear centers is the point P having equal powers with respect to all three circles. It is the intersection of the three radical axes of the circles taken in pairs.

Place triangle ABC in the complex plane so that its circumcircle is the unit circle centered at the origin O. Then $|a| = |b| = |c| = 1$ and the affixes of the points on the Euler line for the triangle are $m(a + b + c)$ for real m; the circumcenter has $m = 0$, the centroid is at $m = 1/3$, the center of the nine point circle is at $m = 1/2$, and $m = 1$ for the orthocenter, the meeting of the three altitudes. Let us suppose the radical center P is on the Euler line, so that $P = m(a + b + c)$. There are complex numbers α and β, with $\alpha + \beta = 1$, such that $d = \alpha b + \beta c$, $e = \alpha c + \beta a$, and $f = \alpha a + \beta b$. Remember that the length of segment OZ, denoted $|z - 0|$ satisfies the equation $|z|^2 = z\bar{z}$. The power of P with respect to the circle centered at D is

$$(PD)^2 - (k\cdot BC)^2 = |m(a + b + c) - (\alpha b + \beta c)|^2 - |k(c - b)|^2$$

$$= [ma + (m - \alpha)b + (m - \beta)c][m\bar{a} + (m - \bar{\alpha})\bar{b} + (m - \bar{\beta})\bar{c}] - k^2(c - b)(\bar{c} - \bar{b}).$$

The powers of P with respect to the circles on E and F will have similar expressions with a, b, and c permuted. Since these expressions are to be equal for all a, b, and c, then they must be symmetric in a, b, and c. Let us look at the coefficients of $a\bar{b}$ and its two permutations $b\bar{c}$ and $c\bar{a}$ in the displayed expression for the power given above. These coefficients must be equal, so we must have

$$m(m - \bar{\alpha}) = (m - \alpha)(m - \bar{\beta}) + k^2 = m(m - \beta). \qquad (1)$$

Similarly, looking at the coefficients of $b\bar{a}$ and its two permutations $c\bar{b}$ and $a\bar{c}$, we obtain the conjugate equations

$$m(m - \alpha) = (m - \bar{\alpha})(m - \beta) + k^2 = m(m - \bar{\beta}).$$

It follows that $\bar{\alpha} = \beta$ and, since $\alpha + \beta = 1$, then $\alpha, \beta = 1/2 \pm ti$ for some real number t and hence the appended triangles must be isosceles. We rewrite the two coefficient equations as

$$m(m - \bar{\alpha}) = (m - \alpha)^2 + k^2 \quad \text{and} \quad m(m - \alpha) = (m - \bar{\alpha})^2 + k^2,$$

which we subtract side for side to get

$$m(\alpha - \bar{\alpha}) = (2m - \bar{\alpha} - \alpha)(\bar{\alpha} - \alpha).$$

Since $\alpha + \bar{\alpha} = 1$, this equation reduces to

$$m = \frac{1}{3}$$

and we see that the radical center P is at the centroid of the triangle. Now substitute back into Equation (1) to get

$$\frac{1}{3}\left(\frac{1}{3} - \bar{\alpha}\right) = \left(\frac{1}{3} - \alpha\right)^2 + k^2,$$

$$\frac{1}{3}\left(\frac{1}{3} - \frac{1}{2} + ti\right) = \left(\frac{1}{3} - \frac{1}{2} - ti\right)^2 + k^2,$$

which reduces to

$$k^2 = t^2 - \frac{1}{12}, \quad \text{so} \quad k = \sqrt{t^2 - \frac{1}{12}}.$$

Hence, for any value of $t > 1/\sqrt{12}$, there is the corresponding value of k that places the radical center of the three appended circles at the centroid of the given triangle.

This problem is a generalization of Problem 408 by the same proposer appearing in vol. 6 (Fall 1978) pp. 554-556 of this JOURNAL. There, points D, E, and F were centers of squares erected on the sides of the triangle.

***920.** [Fall, 1997] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

The *sorted Fibonacci sequence* is produced by starting with the first two terms 1 and 1 and defining each succeeding term as the sum of the prior two terms with the digits sorted into ascending order. Thus we have 1, 1, 2, 3, 5, 8, 13, 12, 25, 37, 26, ... This sequence eventually falls into a repeating cycle.

a) Are there any two initial terms that produce a diverging sequence?

b) How many different repeating cycles can you find?

I. *Partial solution by Mark Evans, Louisville, Kentucky.*

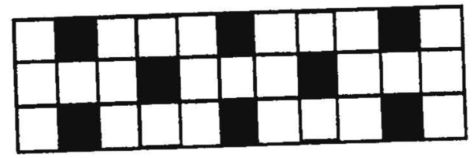
I wrote a program to analyze this problem for all initial pairs where both numbers were less than 101. I found convergence in every instance, with 467 being the greatest number of initial terms before repeating, occurring with the initial pairs (52, 84), (70, 66), (79, 84), and (97, 66). Each converged to the pair (15, 25), whose cycle has 24 terms. In addition, I found 222 pairs which cycle back to themselves. Curiously, these cycles are all of length 3, 8, 9, 24, 48, 96, or 120 only.

Because this mechanism causes the numbers to shrink every time a zero digit is encountered in the sum, I strongly suspect but have not proved that a diverging case does not exist. As the sums get larger, say to 50 or more digits, the probability of there being a zero in the sum is nearly 1. Since some of the repeating cycles involve fairly large numbers, there are many more, perhaps infinitely many, repeating cycles.

Also partially solved by the Proposer. Contact the problems editor if you wish a copy of the 222 pairs and their cycle lengths found by Evans and the 45 pairs found by the proposer.

921. [Fall, 1997] *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*

Place 13 three-digit square numbers in the spaces in the accompanying grid. (The solution is unique.)



Solution by Paul Yiu, Florida Atlantic University, Boca Raton, Florida.

For uniqueness of solution, one must stipulate that the square numbers used are all distinct. Otherwise, there are 14436 solutions. We begin by listing all three-digit squares, the squares of 10 through 31:

100	121	144	169	196	225	256	289	324	361	400
441	484	529	576	625	676	729	784	841	900	961

We see that the tens digit cannot be 1 or 3, the hundreds digit is not 0, and the units digit must be 0, 1, 4, 5, 6, or 9. We shall think of constructing the required grid by extending a hollow 3 by 3 matrix with distinct square numbers along its outer rows and columns. The only possibilities are the following matrices and their transposes:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 4 & * & 0 \\ 4 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 4 \\ 6 & * & 4 \\ 9 & 6 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 & 5 \\ 5 & * & 7 \\ 6 & 7 & 6 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 2 & 4 \\ 6 & * & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} 3 & 2 & 4 \\ 6 & * & 4 \\ 1 & 2 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 3 & 2 & 4 \\ 6 & * & 8 \\ 1 & 4 & 4 \end{pmatrix}, \quad G = \begin{pmatrix} 4 & 4 & 1 \\ 8 & * & 0 \\ 4 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 7 & 2 & 9 \\ 8 & * & 6 \\ 4 & 4 & 1 \end{pmatrix}.$$

By considering the way these matrices are connected by their tens digit entries, we find that A and D can be extended only to the left, G only above, F to the left or below, and H only to the right or below. So none of A, D, F, G, H, and their transposes can be used. Among the remaining B, C, and E and their transposes, only B^T and C can be connected, allowing just the two possibilities

$$\begin{pmatrix} 2 & 2 & 5 & * & 1 & 6 & 9 \\ 5 & * & 7 & 8 & 4 & * & 6 \\ 6 & 7 & 6 & * & 4 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 6 & 9 & * & 2 & 2 & 5 \\ 4 & * & 6 & 2 & 5 & * & 7 \\ 4 & 4 & 1 & * & 6 & 7 & 6 \end{pmatrix}.$$

It is easy to dispose of the first matrix. The first square in the middle row must be 625 since that is the only remaining square ending in 5. Now there is no way to complete the last square in the middle row since both squares beginning with 6 have been used. Only the second matrix remains. If we choose 784 for the last square in the second row, then the first square in the middle row must be 324 or 484. Since 3 cannot be a tens digit and since there is only one remaining square with 4 for its tens digit, namely 841, the grid cannot be completed.

It follows that the last square in the middle row is 729 and the last column is 196. The first square in the middle row is either 784 or 484. Since both squares with tens digit 7 have been used, we cannot have 784. Thus the middle row starts with 484 and the first column is 841. We have the following as the unique solution:

8	■	1	6	9	■	2	2	5	■	1
4	8	4	■	6	2	5	■	7	2	9
1	■	4	4	1	■	6	7	6	■	6

Also solved by Charles D. Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA,* William Bamlet, *Winona State University, Hayfield, MN,* Lisa Blyth, *Wheaton College, IL,* Scott H. Brown, *Auburn University, AL,* Paul S. Bruckman, *Edmonds, WA,* Matt Bussey, *Wheaton College, IL,* Lindsay Coe, *Arkansas Governor's School, Conway,* Cindy Dang, *Arkansas Governor's School, Conway,* Kenneth B. Davenport, *Pittsburgh, PA,* Wilson Davis, *Hendrix College, Conway, AR,* Dierdre DeMeyer, *Wheaton College, IL,* Mark Evans, *Louisville, KY,* James Fells, *Arkansas Governor's School, Conway,* Krista Friesen, *Wheaton College, IL,* Stephen I. Gendler, *Clarion University of Pennsylvania,* Adam Groves, *Wheaton College, IL,* Anthony Hood, *Arkansas Governor's School, Conway,* Tom Huyhn, *Arkansas Governor's School, Conway,* David Iny, *Baltimore MD,* Louis Johnson, *Kansas State University, Manhattan,* Edward John Koslowska, *Angelo State University, San Angelo, TX,* Michael W. Lanstrum, *Independence Community College, KS,* Andrea Lewis, *Wheaton College, IL,* Nicole Mathisen, *Wheaton College, IL,* Carter Price, *Arkansas Governor's School, Conway,* H.-J. Seiffert, *Berlin, Germany,* Sidney Vault, *Arkansas Governor's School, Conway,* Nathan L. Williams, *Wheaton College, IL, and the* Proposer.

922. [Fall, 1997] *Proposed by David Iny, Baltimore, Maryland.*

Suppose that $f(f(x)) = 0$ for all real x. Show that a necessary and sufficient condition that ensures that $f(x) = 0$ for all x is that f be infinitely differentiable on the real line.

Solution by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.

The following function shows that the proposal is incorrect. Let

$$f(x) = e^{-1/x^2} \text{ for } x < 0 \quad \text{and} \quad f(x) = 0 \text{ for } x \geq 0.$$

If $x < 0$, then $f(x) > 0$, so $f(f(x)) = 0$. If $x \geq 0$, then $f(f(x)) = f(0) = 0$. So $f(f(x)) = 0$ for all real x. Now, it is straightforward to show that f is infinitely differentiable on the real line. However, $f(x) \neq 0$ for all $x < 0$.

For the proposal to be correct, the second sentence should be "$f(x) = 0$ for all x if and only if f is (real) analytic on $(-\infty, \infty)$." We show that if f is analytic on $(-\infty, \infty)$ and $f(f(x)) = 0$, then $f(x) = 0$ for all real x. The other half of the proof is trivial.

Assume $f'(y) \neq 0$ for some real y. Since f is analytic, then $f'(x) \neq 0$ on some closed interval $[a, b]$ containing y. Now $f(f(x)) = 0$ for x in $[a, b]$. So

$$f'(f(x)) \cdot f'(x) = 0 \quad \text{and hence} \quad f'(f(x)) = 0$$

for all x in $[a, b]$. Continuing in this fashion we find that $f^{(n)}(f(x)) = 0$ for all x in $[a, b]$ and all positive integers n. Since $f([a, b])$ is an interval $[c, d]$ (because $f'(x) \neq 0$ for all x in $[a, b]$), $f(x) = 0$ for all x in $[c, d]$. Since f is analytic, $f(x) = 0$ on $(-\infty, \infty)$ because the zeros of an analytic function cannot have an accumulation point unless the function itself is identically zero. This contradicts the assumption that $f'(y) \neq 0$ for some real number y. Thus $f'(x) = 0$ for all real numbers x. So $f(x)$ is constant and we have $f(x) = 0$ since $f(f(x)) = 0$.

Also solved by Roger Zarnowski *and* Charles Diminnie, *Angelo State University, San Angelo, TX. Three incorrect solutions were received.*

923. [Fall, 1997] *Proposed by A. Stuparu, Vâlcea, Romania.*

Let $S(n)$ denote the *Smarandache function*: if n is a positive integer, then $S(n) = k$ if k is the smallest nonnegative integer such that $k!$ is divisible by n. Thus $S(1) = 0$, $S(2) = 2$, $S(3) = 3$, and $S(6) = 3$, for example. Prove that the equation $S(x) = p$, where p is a given prime number, has just $2^{p-1} - 1$ solutions between p and $p!$.

I. *Solution by Cecil Rousseau, The University of Memphis, Memphis Tennessee.*

Let $d(n)$ denote the number of divisors of n. The number of solutions of $S(x) = p$ is $d((p-1)!)$ since

$$S(x) = p \quad \text{is equivalent to} \quad p \mid x \text{ and } \frac{x}{p} \,\Big|\, (p-1)!.$$

For example, $S(x) = 5$ had $d(4!) = 8$ solutions, namely 5, 10, 15, 20, 30, 40, 60, and 120, corresponding to the divisors 1, 2, 3, 4, 6, 8, 12, and 24 of 4!.

II. *Solution by Paul S. Bruckman, Edmonds, Washington.*

This problem is incorrectly stated and has an interesting (albeit confusing) history. To the best of this solver's knowledge, the problem first appeared in *The Fibonacci Quarterly*, vol. 32, no. 5 (Nov. 1994), p. 473, as Problem H-490. There, the problem was also incorrectly stated as in this problem, except that the number of solutions was incorrectly given as 2^{p-2} instead of 2^{p-1}.

In a vol. 33, No. 2 (May 1995), p. 187, the problem was reintroduced as H-490 (corrected) to indicate the number of solutions as $d((p-1)!)$, where $d(n)$ is the number of divisors of n. This is indeed the correct formulation of the problem.

The solution to the original H-490 problem was submitted by this solver and published in vol. 33, no. 5 (1995). This solution should be the final word on the subject. Interestingly, the original problem gives the correct values for $p = 2$, 3, and 5 only. This solver sincerely hopes that we can take a stick and beat this problem senseless, to the point where it may not once again resurface in some reincarnated form.

Editorial comment. This proposal was received in April of 1994 by this editor, who deserves 50 lashes with a wet noodle for a misstatement in the number of solutions. I inadvertently omitted a "-1." The proposer had listed the number of solutions as $2^{p-1} - 1$, which, of course, is still incorrect.

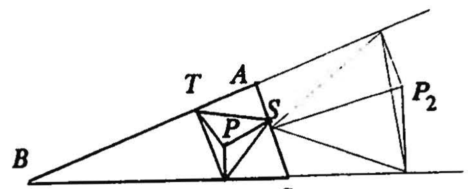
Also solved by Charles D. Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA,* William Chau, *A T & T Laboratories, Middletown, NJ,* Mark Evans, *Louisville, KY,* Stephen I. Gendler, *Clarion University of Pennsylvania, and* H.-J. Seiffert, *Berlin, Germany.*

***924.** [Fall, 1997] *Proposed by George Tsapakidis, Agrino, Greece.*

Find an interior point of a triangle so that its projections on the sides of the triangle are the vertices of an equilateral triangle.

I. *Solution by William H. Peirce, Rangeley, Maine.*

Let ABC be any triangle with sides a, b, and c and let P be any point in the plane of ABC. Then P is uniquely represented by $P = \lambda A + \mu B + \nu C$ where $\lambda + \mu + \nu = 1$ and that P is an interior point if and only if λ, μ, and ν are each positive. Let R, S, and T be the projections of P on sides a, b, and c. The problem requires finding an interior point P of ABC so that triangle RST is equilateral. See the accompanying figure.



Problem 924

By standard analytic geometry we find that R, S, and T separate sides a, b, and c into the following segments:

Side a:	$RC = (b\cos C)\lambda + a\mu$,	$RB = (c\cos B)\lambda + a\nu$,
Side b:	$SA = (c\cos A)\mu + b\nu$,	$SC = (a\cos C)\mu + b\lambda$,
Side c:	$TB = (a\cos B)\nu + c\lambda$,	$TA = (b\cos A)\nu + c\mu$.

Note that the pairs of segments, which are functions of λ, μ, and ν, add respectively to a, b, and c. Negative segments can occur.

Now apply the law of cosines to triangles RCS, SAT, and TBR to get

$$RS^2 = (\sin^2 C)[b^2\lambda^2 + (2ab\cos C)\lambda\mu + a^2\mu^2],$$
$$ST^2 = (\sin^2 A)[c^2\mu^2 + (2bc\cos A)\mu\nu + b^2\nu^2],$$
$$TR^2 = (\sin^2 B)[a^2\nu^2 + (2ca\cos B)\nu\lambda + c^2\lambda^2].$$

Next, set $RS^2 = ST^2$ and $ST^2 = TR^2$ to force RST to be equilateral, obtaining

$$\lambda^2 b^2 \sin^2 C - \nu^2 b^2 \sin^2 A + 2\lambda\mu ab \sin^2 C \cos C - 2\mu\nu bc \sin^2 A \cos A = 0$$

and

$$\mu^2 c^2 \sin^2 A - \lambda^2 c^2 \sin^2 B + 2\mu\nu bc \sin^2 A \cos A - 2\nu\lambda ca \sin^2 B \cos B = 0.$$

(Setting $TR^2 = RS^2$ produces a similar but redundant equation.) Now solve these two equations, along with $\lambda + \mu + \nu = 1$ for λ, μ, and ν. If ABC is itself equilateral, then $\lambda = \mu = \nu = 1/3$ and the one solution point obtained is $P = (A + B + C)/3$, the center of ABC. If ABC is not equilateral, we obtain two the solutions

$$\lambda_{1,2} = \frac{a^2\sin(A\pm 60°)}{D_{1,2}\sin A},\quad \mu_{1,2} = \frac{b^2\sin(B\pm 60°)}{D_{1,2}\sin B},\quad \nu_{1,2} = \frac{c^2\sin(C\pm 60°)}{D_{1,2}\sin C},$$

where

$$D_{1,2} = \frac{1}{2}(a^2 + b^2 + c^2 \pm 4H\sqrt{3})$$

and H is the area of triangle ABC. Solution 1 uses all the plus signs, solution 2 the negative signs. Note that $D_1D_2 = [(a^2 - b^2)^2 + (c^2 - a^2)^2 + (b^2 - c^2)^2]/2$. Now D_1 and D_2 are positive for all ABC except $D_2 = 0$ if ABC is equilateral. It follows that λ_1, μ_1, ν_1, λ_2, μ_2 and ν_2 have the same signs as $\sin(A+60°)$, $\sin(B+60°)$, $\sin(C+60°)$, $\sin(A-60°)$, $\sin(B-60°)$, and $\sin(C-60°)$, respectively.

The following comments complete the solution for all non-equilateral triangles ABC. Recall that P is an interior point of ABC if and only if λ, μ, and ν and all positive.

1. Point P_2 is an exterior point of all triangles ABC because at least one angle, say A, of ABC is less than 60° and therefore $\lambda_2 < 0$. See the accompanying figure. If an angle, say A, equals 60°, then $\lambda_2 = 0$ and P_2 lies on the extension of side a.

2. There are three possibilities for the location of P_1:

a) If each angle of ABC is less than 120°, then λ_1, μ_1, and ν_1 are all positive and P_1 lies inside triangle ABC.

b) If an angle, say A, is 120°, then $\lambda_1 = 0$ and P_1 lies on side a.

c) If an angle, say A, is greater than 120°, then $\lambda_1 < 0$ and P_1 is exterior to triangle ABC. In cases 2a and 2b neither P_1 nor P_2 are interior points, so the stated problem has no solution.

II. *Solution by the Editor.*

Using the notation of Solution I for the points A, B, C, P, R, S, and T, we place triangle ABC in the complex plane with the unit circle centered at the origin as its circumcircle. Thus $|a| = |b| = |c| = 1$, where lower case letters represent the complex affixes of the respective upper case points. We then have

$$r = \frac{1}{2}(b + c + p - bc\bar{p}),\quad s = \frac{1}{2}(c + a + p - ca\bar{p}),$$

and

$$t = \frac{1}{2}(a + b + p - ab\bar{p}).$$

Now RST is equilateral if and only if a 60° rotation, either positive or negative, carries side RS to side RT. So let $z = \operatorname{cis} 60° = \cos 60° + i \sin 60°$. Then z represents a positive 60° rotation and $\bar{z}$ the negative rotation. Setting

$$t - r = z(s - r)$$

and simplifying, we get

$$\bar{p} = \frac{c - a\bar{z} - bz}{bc\bar{z} + caz - ab},$$

and hence

$$p = \frac{\bar{c} - \bar{a}\bar{\bar{z}} - \bar{b}\bar{z}}{\overline{bc\bar{z}} + \overline{caz} - \overline{ab}} = \frac{ab - bcz - ca\bar{z}}{az + b\bar{z} - c},$$

the last form being obtained by multiplying the preceding numerator and denominator by abc. The second point P_2 is obtained by interchanging z and $\bar{z}$ in this last expression.

This derivation, while much briefer than Solution I, does not explain when there is a solution to the given problem.

Also partially solved by Richard I. Hess, *Rancho Palos Verdes, CA. One incorrect solution was also received.*

925. [Fall, 1997] *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*

Given positive integers s and c and any integer k such that $0 \le k \le s$, prove that

$$(-1)^k \sum_{j=k}^{s} \binom{c+j-1}{c-1}\binom{j}{k} = \sum_{j=0}^{k} (-1)^j \binom{s+c}{s+j-k}\binom{s+j-k}{j}.$$

I. *Solution by H.-J. Seiffert, Berlin, Germany.*

The identities

$$\sum_{j=m}^{n} \binom{j}{m} = \binom{n+1}{m+1}, \quad m, n \in N_0, \quad n \ge m, \tag{1}$$

and

$$\sum_{j=0}^{k} (-1)^j \binom{m}{j} = (-1)^k \binom{m-1}{k}, \quad k, m, n \in N_0, \quad m > k, \tag{2}$$

are well known and easily proved. In fact, (1) can be verified by an easy induction on n, and (2) by induction on m.

Let L and R denote the left and right expressions in the given equation, respectively. Since

$$\binom{c+j-1}{c-1}\binom{j}{k} = \binom{c+k-1}{c-1}\binom{c+j-1}{c+k-1}, \quad j = k, \ldots, s,$$

from (1) with $m = c + k - 1$ and $n = s + c - 1$, after a suitable reindexing, we find

$$L = (-1)^k \binom{c+k-1}{c-1}\binom{s+c}{c+k}. \tag{3}$$

Using

$$\binom{s+c}{s+j-k}\binom{s+j-k}{j} = \binom{s+c}{c+k}\binom{c+k}{j}, \quad j = 0, \ldots, k,$$

then (2) with $m = c + k$ gives

$$R = (-1)^k \binom{s+c}{c+k}\binom{c+k-1}{k}. \tag{4}$$

From (3) and (4) it is clear that $L = R$.

II. *Comment by the Proposer.*

Consider a series of independent experiments where the result of each experiment is success or failure and the probability of success is p for each experiment. Let P_1 be the probability of getting exactly c successes in r or fewer tries, and let P_2 be the probability of getting at least c successes in exactly r tries. It is readily seen that

$$P_1 = \sum_{y=c}^{r} \binom{y-1}{c-1} p^c (1-p)^{y-c} = \sum_{y=c}^{r}\sum_{j=0}^{y-c} (-1)^j \binom{y-1}{c-1}\binom{y-c}{j} p^{j+c},$$

whereas

$$P_2 = \sum_{y=c}^{r} \binom{r}{y} p^y (1-p)^{r-y} = \sum_{y=c}^{r}\sum_{j=0}^{r-y} (-1)^j \binom{r}{y}\binom{r-y}{j} p^{j+y}.$$

The interesting and somewhat surprising fact is that $P_1 = P_2$. Dividing P_1 and P_2 by p^c and equating coefficients of p^k in the resulting expressions yields

$$(-1)^k \sum_{y=c+k}^{r} \binom{y-1}{c-1}\binom{y-c}{k} = \sum_{j=0}^{k} (-1)^j \binom{r}{k+c-j}\binom{r+j-k-c}{j},$$

where $0 \le k \le r - c$. Setting $j = y - c$ in the left expression and $s = r - c$ in both yields the proposed problem.

Also solved by Paul S. Bruckman, *Edmonds, WA, and* Cecil Rousseau, *The University of Memphis, TN.*

926. [Fall, 1997] *Proposed by Tom Moore, Bridgewater State College, Bridgewater, Massachusetts.*

Students were asked the question, "How many times is $x := x + 1$ executed in the following nested loop?

```
For i = 2 to n
    For j = 1 to ⌊i/2⌋
        x:= x + 1
    Next j
Next i
```

Discover which of the following ten actual student answers are correct, where $\lfloor\ \rfloor$ is the floor function and $\lceil\ \rceil$ is the ceiling function (so that $\lfloor\pi\rfloor = 3$ and $\lceil\pi\rceil = 4$):

$$a(n) = \begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \frac{n^2 - 1}{4}, & n \text{ odd.} \end{cases} \qquad b(n) = \begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \left\lfloor\frac{n+1}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor, & n \text{ odd.} \end{cases}$$

$$c(n) = \begin{cases} \frac{n^2}{4}, & n \text{ even} \\ \left(\frac{n-1}{2}\right)\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right), & n \text{ odd.} \end{cases} \qquad d(n) = \left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n+1}{2}\right\rfloor.$$

$$e(n) = \left\lfloor\frac{n}{2}\right\rfloor^2 + \left\lfloor\frac{n}{2}\right\rfloor. \qquad f(n) = \left\lfloor\frac{n}{2}\right\rfloor\left\lceil\frac{n}{2}\right\rceil.$$

$$g(n) = \left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor + [n \pmod 2]\right). \qquad h(n) = \left\lfloor\frac{n^2+2}{4}\right\rfloor.$$

$$i(n) = \left\lfloor\frac{n^2}{4}\right\rfloor. \qquad j(n) = \sum_{k=2}^{n}\left\lfloor\frac{k}{2}\right\rfloor.$$

Solution by Sean Williams, student, Alma College, Alma, Michigan.

Solution $e(n)$ is false; all others are correct. By executing the given loop structure for $n = 2, 3, ..., 10$, we find that $x := x + 1$ is executed 1, 2, 4, 6, 9, 12, 16, 20, 25 times, respectively. It is seen that the outer loop is executed $n - 1$ times and the inner loop $\lfloor k/2\rfloor$ for each k, $2 \le k \le n$. This result is summarized by equation $j(n)$:

$$j(n) = \sum_{k=2}^{n}\left\lfloor\frac{k}{2}\right\rfloor = 1 + 1 + 2 + 2 + 3 + 3 + \cdots + \left\lfloor\frac{n}{2}\right\rfloor. \qquad (1)$$

Recall that

$$\sum_{i=1}^{k} i = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

If n is odd, sum (1) is

$$1 + 1 + 2 + 2 + \cdots + \frac{n-1}{2} + \frac{n-1}{2} = \frac{n-1}{2}\cdot\frac{n+1}{2}.$$

If n is even, the sum is

$$1 + 1 + 2 + 2 + \cdots + \frac{n-2}{2} + \frac{n-2}{2} + \frac{n}{2} = \frac{n-2}{2}\cdot\frac{n}{2} + \frac{n}{2} = \frac{n^2}{4}.$$

Each answer can now be compared with these results, considering odd and even cases when necessary. All are seen to be correct except $e(n)$, which is false for even n, since, for example, $e(6) = 12$ but $j(6) = 9$.

Also solved by Charles D. Ashbacher, *Charles Ashbacher Technologies, Hiawatha, IA,* Paul S. Bruckman, *Edmonds, WA,* William Chau, *A T & T Laboratories, Middletown, NJ,* Russell Euler *and* Jawad Sadek, *Northwest Missouri State University, Maryville,* Mark Evans, *Louisville, KY,* David Fullerton, *Alma College, MI,* Richard I. Hess, *Rancho Palos Verdes, CA,* Carl Libis, *University of Alabama, Tuscaloosa,* H.-J. Seiffert, *Berlin, Germany,* Mike Slater, *Alma College, MI, and the* Proposer.

Corrections

The editor apologizes for inadvertently omitting the following names from the also-solver lists for the following problems: Miguel Amengual Covas, *Cala Figuera, Mallorca, Spain*, problems 890, 891, 896, 898, and 900; Yoshinobu Murayoshi, *Okinawa, Japan*, problem 854; Kenneth M, Wilke, *Topeka KS*, problems 849, 852, 854, and 857; and Rex H. Wu, *Brooklyn, NY*, problem 860. One incorrect solution to problem 860 was also received.

Murray Klamkin pointed out errors in his solution to Problem 853, vol. 10 (Spring 1996) p.330. In the inequality beginning with $S(m, n)$, displayed in the center of the page, the two exponents should be $(2n - 1)/n$ and $1/(2n)$, instead of $(n - 1)/n$ and $1/n$.

Problems Website Announcement

A new website called MathPro Online has been developed by MathPro Press, founded by Stanley Rabinowitz. It allows users to search electronically its nearly 23,000 problems from 42 journals and 22 contests. Its URL is

http://problems.math.umr.edu

and was generously donated by Leon M. Hall, Professor and Director of Graduate Studies in the Department of Mathematics and Statistics at the University of Missouri - Rolla.

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CHAPTER REPORTS

Professor Cathy Talley reported that the **TEXAS ZETA Chapter** (Angelo State University) co-sponsored seven mathematics forums during the year. Three student members of the chapter had solutions published in the **Journal**.

Professor Joanne Snow reported that the **INDIANA EPSILON Chapter** (Saint Mary's College) was addressed by **Underwood Dudley** (DePauw University) at the department's annual Open House. The chapter performed various service activities during the year.

Dr. David Sutherland reported that the **ARKANSAS BETA Chapter** (Hendrix College) was addressed by several guest speakers during the year. Their Undergraduate Research Program was very active. Six members of the chapter presented papers. Several students received awards at the Honors Convocation on May 20.

Professor David Vella reported that the **NEW YORK ALPHA THETA Chapter** (Skidmore College) was addressed by **William Swicker** (Union College) at the first official event of the new chapter. The chapter has been busy establishing Bylaws.