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PROBLEM DEPARTMENT

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This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at `sjm1@williams.edu`*; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number and write down the full statement of the problem. Solutions identified as by students are given preference.*

Problems for Solution.

1288. *Proposed by Gabriel Prajitura, Mathematics Department, SUNY Brockport.*

A term a_k of a sequence $\{a_n\}$ is called a *local extreme* if either $a_{k-1} \leq a_k \geq a_{k+1}$ or $a_{k-1} \geq a_k \leq a_{k+1}$. (a) If a sequence has infinitely many local extreme terms prove that the sequence is convergent if and only if the subsequence of all local extreme terms is convergent. (b) Show that Part (a) is no longer true if in the definition of a local extreme $\leq$ and $\geq$ are replaced by $<$ and $>$ respectively.

1289. *Proposed by Mike Pinter, Belmont University, Nashville, TN.*

In honor of the centennial of Pi Mu Epsilon, solve in base 16

$$\begin{array}{r} PMEMATH \\ +\ SOCIETY \\ \hline HUNDRED \end{array}$$

(note there are 15 different letters).

1290. *Proposed by Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania.*

Consider a set of five distinct positive real numbers such that if we take all products of pairs of these numbers then only seven distinct numbers are formed. Thus if the numbers are $0 < x_1 < x_2 < x_3 < x_4 < x_5$, if we look at the set formed from all products $x_i x_j$ with $i \neq j$ then there are only seven distinct numbers. Prove the x_i's form an geometric progression; in other words, there is an r such that $x_{i+1} = r x_i$ for $i \in \{1, 2, 3, 4\}$.

1291. *Chirita Marcel, Bucharest, Romania.*

Given $x_1, x_2, x_3, x_4, x_5, x_6 \in (0, \infty)$ such that

$$\frac{1}{x_1 + x_2} + \frac{1}{x_3 + x_4} + \frac{1}{x_5 + x_6} \ = \ 1,$$

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prove that

$$\left(\sum_{i=1}^{6} x_i\right)^2 \left(\sum_{i=1}^{6} x_i + 9\right) \ \geq \ 54(x_1+x_2)(x_3+x_4)(x_5+x_6).$$

1292. *Proposed by Moti Levy, Rehovot, Israel.*

Let $f(x)$ and $f^2(x)$ be Riemann-integrable functions on $[0,1]$, and let $g(x)$ be a twice-differentiable function on $[0,1]$ such that $g(0)=1$.
a) Show that

$$\lim_{n\to\infty}\prod_{k=1}^{n} g\left(\frac{1}{n} f\left(\frac{k}{n}\right)\right) \ = \ \exp\left(g'(0)\int_0^1 f(x)\, dx\right).$$

b) Find a suitable choice of the functions $f(x)$ and $g(x)$ to solve Problem 1892 from *Mathematics Magazine* (proposed by Jose Luis Dıaz-Barrero):

$$\lim_{n\to\infty}\frac{1}{n^n}\prod_{k=1}^{n}\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}} \ = \frac{4}{e}.$$

1293. *Proposed by Steven J. Miller, Williams College.*

The following is from the 2010 Green Chicken Contest between Middlebury and Williams.

Every year Middlebury and Williams have a math competition among their students, with the winning team getting to keep the infamous Green Chicken till the following year; see
`http://web.williams.edu/Mathematics/sjmiller/public_html/greenchicken/index.htm`
for pictures and additional history and problems. The following is a modification of a problem from 2010.

Instead of taking a math contest, Middlebury and Williams decide to settle who gets the Green Chicken by playing the following game. Consider the first one million positive integers. Player A's goal is to choose 10,000 of these numbers such that at the end of the choosing procedure there are at least 20 pairs of chosen integers with the same positive difference (for example, (12,39), (39,66) and (101,128) count as three pairs with a difference of 27). A turn consists of Player A choosing 10 numbers, and then Player B moving up to 10 of *any* number chosen to any unchosen number. We keep playing until A has chosen 10,000 numbers, allowing B to get its final turn. Determine which player has a winning strategy, and prove your claim.

Solutions: Spring 2014.

1283. *Proposed by D. Andrica, E. Ionascu and R. Stephens, Columbus State University, Columbus, GA.*

Let k and n be positive integers. For the set $S_{k,n} = \{1^k, 2^k, \ldots, n^k\}$, consider the question "Can $S_{k,n}$ be partitioned into two nonempty subsets, each having the same sum?" Let $P_{k,n}$ be the number of ways to partition $S_{k,n}$ in this manner. For example

- $P_{1,3} = 1$ with $S_{1,3} = \{1,2\} \cup \{3\}$.
- $P_{1,4} = 1$ with $S_{1,4} = \{1,4\} \cup \{2,3\}$.
- $P_{1,5} = P_{1,6} = 0$.

- $P_{1,7} = 4$ with $S_{1,7} = \{1, 2, 4, 7\} \cup \{3, 5, 6\}$, or $\{1, 6, 7\} \cup \{2, 3, 4, 5\}$, or $\{2, 5, 7\} \cup \{1, 3, 4, 6\}$, or $\{3, 4, 7\} \cup \{1, 2, 5, 6\}$.
- $P_{2,n} = 0$ for $n \leq 6$.
- $P_{2,7} = 1$ with $S_{2,7} = \{1^2, 2^2, 4^2, 7^2\} \cup \{3^2, 5^2, 6^2\}$.

Question 1: Find the smallest value of n for which $P_{2,n} > 1$. Explain your answer and identify any technology used.

Question 2: Find a general "formula" for determining P, the number of ways that the finite set $S = \{\theta_1, \theta_2 \ldots, \theta_n\}$ of integers can be partitioned int tow nonempty subsets, each having the same sum.

Solution by the **Missouri State University Problem Solving Group**, *Department of Mathematics, Missouri State University, Springfield, MO.*

We answer the second question first and use that method to answer the first. If

$$A = \sum_{i=1}^{n} \theta_i$$

is odd, there are no solutions (since, in that case, it is impossible to split the set into two subsets with equal sums). If A is even, then the number of solutions is half of the coefficient of

$$x^{\sum_{i=1}^{n} \theta_i/2}$$

in the expansion of

$$B = \prod_{i=1}^{n} (1 + x^{\theta_i}).$$

From the theory of (ordinary) generating functions, it is clear that the coefficient of x^k in B is the number of ways that the θ_i's can be chosen so that their sum is k. In our case,

$$k = \left(\sum_{i=1}^{n} \theta_i \right) /2$$

and the sums come in complementary pairs.

For this problem, we use *Mathematica* to compute the coefficient of $x^{n(n+1)(2n+1)/12}$ in the expansion of $\prod_{i=1}^{n}(1 + x^{i^2})$, when $n = 8, 11, 12, 15, 16, 19, 20, \ldots$ (these are the only values of n for which $n(n + 1)(2n + 1)/12$ is an integer). Doing so, we find that

$$\begin{aligned} P_{2,8} &= 1 \\ P_{2,11} &= 1 \\ P_{2,12} &= 5 \\ P_{2,15} &= 43 \\ P_{2,16} &= 57 \\ P_{2,19} &= 239 \\ P_{2,20} &= 430. \end{aligned}$$

For completeness, we list the partitions for $n = 8$, 11, and 12:

$$\begin{aligned}
S_{2,8} &= \{1^2, 4^2, 6^2, 7^2\} \cup \{2^2, 3^2, 5^2, 8^2\}. \\
S_{2,11} &= \{1^2, 3^2, 4^2, 5^2, 9^2, 11^2\} \cup \{2^2, 6^2, 7^2, 8^2, 10^2\}. \\
S_{2,12} &= \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 11^2\} \cup \{9^2, 10^2, 12^2\} \text{ or} \\
&= \{1^2, 2^2, 3^2, 4^2, 5^2, 7^2, 10^2, 11^2\} \cup \{6^2, 8^2, 9^2, 12^2\} \text{ or} \\
&= \{1^2, 3^2, 4^2, 5^2, 7^2, 9^2, 12^2\} \cup \{2^2, 6^2, 8^2, 10^2, 11^2\} \text{ or} \\
&= \{1^2, 3^2, 7^2, 8^2, 9^2, 11^2\} \cup \{2^2, 4^2, 5^2, 6^2, 10^2, 12^2\} \text{ or} \\
&= \{1^2, 4^2, 8^2, 10^2, 12^2\} \cup \{2^2, 3^2, 5^2, 6^2, 7^2, 9^2, 11^2\}.
\end{aligned}$$

The answer to the original question is $n = 12$.

1285. *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.*

Find all solutions in integers a and b to the equation

$$(a + 3)(a^2 + 3) = b^2 + 7.$$

Solution by **David E. Manes***, SUNY College at Oneonta, Oneonta, NY.*

The integral solutions (a, b) of the equation are $(-2, 0)$, $(-1, \pm 1)$ and $(1, \pm 3)$. To see this, observe that the equation $(a + 3)(a^2 + 3) = b^2 + 7$ is equivalent to $b^2 - (a + 1)^3 = 1$, and so is an equation of the form $x^2 - y^3 = 1$. Euler found that the only integer solutions (x, y) of this equation are $(0, -1)$, $(\pm 1, 0)$ and $(\pm 3, 2)$. For a proof that these are the only rational solutions as well, see Corollary 2a in W. Sierpinski's *Elementary Theory of Numbers*, Hafner, 1964, p. 81. Hence the only integer solutions (a, b) of $b^2 - (a + 1)^3 = 1$ are $(-2, 0)$, $(-1, \pm 1)$ and $(1, \pm 3)$.

Note from the editor: Many people came close to solving this directly. Several groups found that it could be rewritten as asking for when a square is one more than a cube. People frequently broke this up into cases, but often missed a case or assumed a related equation had no integral solutions. The square-cube difference is a realization of a more general problem: given integers $m, n \geq 2$ when can we have $x^m - y^n = 1$ solvable in the integers? It was conjectured by Catalan that the only non-trivial solution is $3^2 - 2^3 = 1$ (as $1^m - 0^n = 1$ always works). Not surprisingly this result became known as Catalan's conjecture (as noted by Kipp Johnson, Valley Catholic School, Beaverton, Oregon, who used this result to solve the problem), and was proved by Preda Mihăilescu in 2004. For a outline of the proof, see the following paper from the Bulletin of the AMS:

[1] Tauno Metsänkylä, *Catalan's conjecture: another old Diophantine problem solved*, Bulletin of the American Mathematical Society **41**, no. 1, 43–57) 2004.
`http://www.ams.org/journals/bull/2004-41-01/S0273-0979-03-00993-5/S0273-0979-03-00993-5.pdf`

1286. *Proposed by Gabriel Prajitura, Mathematics Department, SUNY Brockport.*

Find an arithmetic progression of natural numbers such that the distance from any term of the progression to any perfect square is at least 7.

Solution by the **Missouri State University Problem Solving Group***, Department of Mathematics, Missouri State University, Springfield, MO.*

We claim that the arithmetic progression $\{120n + 113\}$ has the desired property. To see this, we note that none of the integers in the range from 106 to 119 inclusive are squares modulo 120 (since in that range only 108, 112, 113, and 116 are squares modulo 8, the first three of these are not squares modulo 5, and the last is not a square modulo 3). Therefore none of the numbers $120n+106$ through $120n+119$ can be perfect squares. Since the nearest possible squares to $120n+113$ are $120n+105$ and $120n+120$, the result follows.

More generally, we claim that for any natural number d there is an arithmetic progression such that the distance from any of its terms to any perfect square is at least d. Arguing as above, it will suffice to show that there are natural numbers n such that there are $2d-1$ consecutive quadratic non-residues modulo n. We will prove by induction that for any natural number N, there is an n such that $-1,-2,\ldots,-N$ are all quadratic non-residues modulo n. If N=1, we take $n=3$. Suppose that $-1,-2,\ldots,-N$ are all quadratic non-residues modulo n. Denote the square-free part of $N+1$ by $\operatorname{sqp}(N+1)$ and let $q_i, i=1,\ldots,k$ be the prime factors of $\operatorname{sqp}(N+1)$ that are congruent to 3 modulo 4 and $r_i, i=1,\ldots,\ell$ those that are congruent to 1 modulo 4. Therefore

$$\operatorname{sqp}(N+1)=2^{\epsilon}\prod_{i=1}^{k}q_i\prod_{i=1}^{\ell}r_i,$$

where $\epsilon=0$ or 1. Use the Chinese Remainder Theorem to choose a prime p such that

$$p\equiv -1 \bmod 8\prod_{i=1}^{k}q_i \text{ and } p\equiv 1 \bmod \prod_{i=1}^{\ell}r_i$$

(there are infinitely many such primes by Dirichlet's theorem). Using basic properties of the Legendre symbol we have $\left(\frac{-1}{p}\right)=-1$, since $p\equiv 3 \bmod 4$, and $\left(\frac{2^{\epsilon}}{p}\right)=1$ (regardless of the value of ϵ), since $p\equiv 7 \bmod 8$. By Quadratic Reciprocity

$$\begin{aligned}\left(\frac{q_i}{p}\right) &= \left(\frac{p}{q_i}\right)(-1)^{(p-1)(q_i-1)/4}\\ &= \left(\frac{-1}{q_i}\right)(-1)\ \text{[since } p\equiv -1 \bmod q_i \text{ and since } p\equiv q_i\equiv 3 \bmod 4]\\ &= (-1)(-1)\ \text{[since } q_i\equiv 3 \bmod 4]\\ &= 1,\end{aligned}$$

and

$$\begin{aligned}\left(\frac{r_i}{p}\right) &= \left(\frac{p}{r_i}\right)(-1)^{(p-1)(r_i-1)/4}\\ &= \left(\frac{1}{r_i}\right)(1)\ \text{[since } p\equiv 1 \bmod r_i \text{ and since } r_i\equiv 1 \bmod 4 \text{ and } p-1 \text{ is even]}\\ &= 1.\end{aligned}$$

Combining the results above

$$\begin{aligned}\left(\frac{-(N+1)}{p}\right) &= \left(\frac{-\operatorname{sqp}(N+1)}{p}\right)\\ &= \left(\frac{-1}{p}\right)\left(\frac{2^{\epsilon}}{p}\right)\prod_{i=1}^{k}\left(\frac{q_i}{p}\right)\prod_{i=1}^{\ell}\left(\frac{r_i}{p}\right)\\ &= (-1)(1)(1)(1)\\ &= -1.\end{aligned}$$

We claim that $-1, -2, \ldots, -N$ are quadratic non-residues modulo pn. If $-k$ were a quadratic residue mod pn, then there would be an x such that $x^2 \equiv -k \bmod pn$, but this would force $x^2 \equiv -k \bmod n$, which is impossible. Similarly, $-(N+1)$ is a quadratic non-residue modulo pn, since it is modulo p (regardless of its status modulo n). Therefore $-1, -2, \ldots, -(N+1)$ are all quadratic non-residues modulo pn and the result follows by induction.

This problem was also solved and generalized by **Brian D. Beasley**, *Presbyterian College, Clinton, SC, shortly after the first solution was received; we give his solution below (but in the interest of space not his generalization).*

It was also solved by **Kipp Johnson**, *Valley Catholic School, Beaverton, Oregon. His solution is similar but instead uses the progression* $\{192n + 184 : n \geq 0\}$. *Abhay Malik from the Episcopal Academy 9th Grade solved it by looking at* $\{5600n + 56 : n \geq 0\}$.

To solve the original problem, we first calculate the set of squares modulo 144, obtaining $\{0, 1, 4, 9, 16, 25, 36, 49, 52, 64, 73, 81, 97, 100, 112, 121\}$. Since there is a gap of length 23 between the squares 121 and 0, we have two arithmetic progressions of natural numbers such that the distance from any term of either progression to any square is at least 11: They are $\{144n + 132 : n \geq 0\}$ and $\{144n + 133 : n \geq 0\}$.