

# PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2014

STEVEN J. MILLER (EDITOR)

## 1. PROBLEMS: FALL 2014

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (\*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at [sjm1@williams.edu](mailto:sjm1@williams.edu); proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.

**#1294:** *Chirita Marcel, Bucharest, Romania.*

Let  $f$  be a differentiable function such that, for some  $a, b$  satisfying  $0 < a < b < 1$  we have

$$\int_0^a f(x)dx = \int_b^1 f(x)dx = 0.$$

Prove that

$$\left| \int_0^1 f(x)dx \right| \leq \frac{1-a+b}{4} \sup_{x \in (0,1)} |f'(x)|.$$

**#1295:** *Proposed by Moti Levy, Rehovot, Israel.*

This problem is related to **Problem 1889** proposed by Gary Gordon and Peter McGrath in *Mathematics Magazine*. For every positive integer  $k$ , consider the series

$$\begin{aligned} S_k = & \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \right) - \left( \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} \right) \\ & + \left( \frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{3k} \right) - \left( \frac{1}{3k+1} + \frac{1}{3k+2} + \cdots + \frac{1}{4k} \right) \pm \cdots \end{aligned}$$

(a) Show that  $H_k > S_k > \alpha H_k$  for some  $\alpha$ ,  $0 < \alpha < 1$ .  $H_k = \sum_{m=1}^k \frac{1}{m}$  is the harmonic series.

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*Date:* November 3, 2014.

(b) Prove that  $\lim_{k \rightarrow \infty} (H_k - S_k) = \ln \frac{\pi}{2}$ .

(c) Find a closed form of  $S_k$  (with finite number of summands).

Check your solution by evaluating  $S_8 = \left( \frac{1}{16} + \frac{\sqrt{2}}{8} + \frac{1}{4} \sqrt{\sqrt{2} + 2} \right) \pi + \frac{1}{8} \ln 2$ .

(d) Show that  $\lim_{k \rightarrow \infty} \left( \sum_{m=1}^{k-1} \frac{\pi}{2k \sin \frac{m\pi}{k}} - \ln \frac{2k}{\pi} \right) = \gamma$ , where  $\gamma$  is the Euler-Mascheroni constant.

**#1296:** Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA 02325

In this problem all variables represent positive integers. (i) Prove that  $a^2 + b^2 = ab$  is impossible. (ii) If  $a, b, c$  satisfy  $a^2 + b^2 + c^2 = abc$ , then prove that  $27|abc$ . (iii) If  $a, b, c, d$  satisfy  $a^2 + b^2 + c^2 + d^2 = abcd$ , then prove that  $16|abcd$ . (iv\*) Prove or disprove: if  $a, b, c, d, e$  satisfy  $a^2 + b^2 + c^2 + d^2 = abcde$ , then  $9|abcde$ . More generally, what might be true for the sum of  $n$  squares?

**1297:** Proposed by Ben Klein, Davidson College, Davidson, NC.

Suppose that  $p(z)$  is a cubic polynomial over the complex numbers with distinct roots,  $a_1, a_2, a_3$ .

(1) Assuming  $p'(a_1) = p'(a_3)$ , express  $a_2$  as a function of  $a_1$  and  $a_3$ , and express  $p'(a_2)$  as a function of  $p'(a_1)$ .

(2) Assuming  $p'(a_1) \neq p'(a_3)$ , express  $p'(a_2)$  as a function of  $p'(a_1)$  and  $p'(a_3)$ .

**1298:** Proposed by Arthur L. Holshouser, Charlotte, NC.

For all positive integers  $i$ , let  $x_i = -a_i + a_{i+1} + a_{i+2}$ , where  $a_i = a_{i+3}$ . If  $\bar{t}, a_1, a_2, a_3$  are given real numbers with  $a_1, a_2, a_3$  distinct and nonzero and  $x_1, x_2, x_3$  are nonzero, show that there exists a unique real number  $t$  of the form  $t = \frac{\bar{t}-1}{\bar{t}+f(a,b,c)}$  such that

$$\frac{a_i x_i + a_{i+1} x_{i+1} + t a_{i+2} x_{i+2}}{a_i x_i + t a_{i+1} x_{i+1} + a_{i+2} x_{i+2}} = \frac{a_{i+1}(a_{i+1} + a_{i+2})x_{i+1} + a_i(a_i + a_{i+2})x_i + \bar{t} a_i a_{i+1}(x_i + x_{i+1})}{a_{i+2}(a_{i+1} + a_{i+2})x_{i+2} + a_i(a_i + a_{i+1})x_i + \bar{t} a_i a_{i+2}(x_i + x_{i+2})}$$

for  $i \in \{1, 2, 3\}$ .

**1299:** Proposed by Steven J. Miller, Williams College, Williamstown, MA 01267

**The final problem is from the 2014 Middlebury - Williams Green Chicken Contest; as that takes place on November 15th, the problem cannot be posted until shortly after that. Please check back later.**

## 2. SOLUTIONS

Note: A correct solution to problem #1283 from Mark Evans of Louisville, KY arrived after the previous issue went to press.

**#1287:** *Proposed by David Rhee, Massachusetts Institute of Technology, Boston, MA.*

*Amy and Peter are sharing a cake. Amy will cut it into two pieces. Peter then cuts one of the pieces into two. This is followed by a second cut by Amy and a second cut by Peter, so that there will be five pieces, of sizes  $0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ , with  $a_1 + a_2 + a_3 + a_4 + a_5 = 1$ . Amy will get the three pieces of sizes  $a_1$ ,  $a_3$  and  $a_5$ , while Peter will get the remaining two pieces. What is the maximum amount of the cake Amy can get?*

*Solution by **Andy Liu, Robert Barrington Leigh, David Rhee and Yun Hao Fu.** This problem was also solved by Mark Evans of Louisville, KY, who argued similarly.*

First we prove that Peter can always get  $\frac{2}{5}$  of the cake. Suppose Amy cuts the cake into two pieces of sizes  $x$  and  $1 - x$ , where  $0 \leq x \leq \frac{1}{2}$ . There are three cases.

**Case 1.**  $\frac{2}{5} \leq x \leq \frac{1}{2}$ .

Peter will cut  $1 - x$  into  $x$  and  $1 - 2x$ . Now the three pieces are of sizes  $1 - 2x < x = x$ . If Amy does not cut either  $x$ , neither will Peter. Peter will then be sure of getting  $x$  plus a second piece, and  $x \geq \frac{2}{5}$ . If Amy cuts one of  $x$ , Peter will cut the other  $x$  in the same proportions. Peter will get two pieces which add up to  $x \geq \frac{2}{5}$ .

**Case 2.**  $\frac{1}{5} \leq x < \frac{2}{5}$ .

Peter will cut  $x$  into  $x - \frac{1}{5}$  and  $\frac{1}{5}$ . Now the three pieces are of sizes  $x - \frac{1}{5} < \frac{1}{5} < 1 - x$ . If Amy does not cut  $1 - x$ , Peter will cut this in halves. The second smallest piece cannot be less than  $\frac{1}{2}(x - \frac{1}{5})$ , so Peter will get at least  $\frac{1-x}{2} + \frac{1}{2}(x - \frac{1}{5}) = \frac{2}{5}$ . Suppose Amy cuts  $1 - x$  into  $y$  and  $1 - x - y$ , where  $0 \leq y \leq \frac{1-x}{2}$ . Then Peter will cut  $1 - x - y$  into  $\frac{2}{5} - y$  and  $\frac{3}{5} - x$ . Now  $y + (\frac{2}{5} - y) = \frac{2}{5} = (x - \frac{1}{5}) + (\frac{3}{5} - x)$ . Thus Peter will get two pieces which add up to  $\frac{2}{5}$ .

**Case 3.**  $0 \leq x < \frac{1}{5}$ .

Peter will cut  $1 - x$  into  $\frac{1}{5}$  and  $\frac{4}{5} - x$ . The situation is exactly the same as in Case 2.

We now prove that Amy can always get  $\frac{3}{5}$  of the cake. She can start by cutting the cake into two pieces of sizes  $\frac{2}{5}$  and  $\frac{3}{5}$ . There are two cases.

**Case 1.** Peter cuts  $\frac{2}{5}$  into  $x$  and  $\frac{2}{5} - x$ , where  $0 \leq x \leq \frac{1}{5}$ .

Amy will cut  $\frac{3}{5}$  into  $x$  and  $\frac{3}{5} - x$ . Now the four pieces are of sizes  $x = x \leq \frac{2}{5} - x < \frac{3}{5} - x$ . No matter what Peter does, the size of the second largest piece is at most  $\frac{2}{5} - x$  and the size of the fourth largest piece is at most  $x$ . Hence Peter gets at most  $(\frac{2}{5} - x) + x = \frac{2}{5}$ .

**Case 2.** Peter cuts  $\frac{3}{5}$  into  $x$  and  $\frac{3}{5} - x$ , where  $0 \leq x \leq \frac{3}{10}$ .

If  $0 \leq x \leq \frac{1}{5}$ , Amy will cut  $\frac{2}{5}$  into  $x$  and  $\frac{2}{5} - x$ , and the situation is exactly the same as in Case 1. Hence we may assume that  $\frac{1}{5} < x \leq \frac{3}{10}$ . Amy will cut  $\frac{3}{5} - x$  into  $\frac{1}{5}$  and  $\frac{2}{5} - x$ . Now the four pieces are of sizes  $\frac{2}{5} - x < \frac{1}{5} < x < \frac{2}{5}$ . There are four subcases.

**Subcase 2(a).** Peter cuts  $\frac{2}{5}$  into  $y$  and  $\frac{2}{5} - y$ , where  $0 \leq y \leq \frac{1}{5}$ .

Since  $y + (\frac{2}{5} - y) = \frac{2}{5} = x + (\frac{2}{5} - x)$ , Peter will get two pieces which add up to  $\frac{2}{5}$ .

**Subcase 2(b).** Peter cuts  $x$ .

If  $\frac{1}{5}$  remains the third largest piece, Amy will get at least  $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$ . If it becomes the second largest piece, Peter gets at most  $\frac{1}{5} + \frac{1}{5} = \frac{2}{5}$ .

**Subcase 2(c).** Peter cuts  $\frac{1}{5}$  into  $y$  and  $\frac{1}{5} - y$ , where  $0 \leq y \leq \frac{1}{10}$ .

Since  $\frac{2}{5} - x \geq y$ , the second smallest piece is at most  $\frac{2}{5} - x$ . Hence Peter gets at most  $(\frac{2}{5} - x) + x = \frac{2}{5}$ .

**Subcase 2(d).** Peter cuts  $\frac{2}{5} - x$ .

Amy will get at least  $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$ .

**Problem 1288:** *Proposed by Gabriel Prajitura, Mathematics Department SUNY Brockport.*

A term  $a_k$  of a sequence  $\{a_n\}$  is called a local extreme if either  $a_{k-1} \leq a_k \geq a_{k+1}$  or  $a_{k-1} \leq a_k \geq a_{k+1}$ . (a) If a sequence has infinitely many local extreme terms prove that the sequence is convergent if and only if the subsequence of all local extreme terms is convergent. (b) Show that Part (a) is no longer true if in the definition of a local extreme  $\leq$  and  $\geq$  are replaced by  $<$  and  $>$  respectively.

*Solution by Eugen J. Ionascu, Department of Mathematics, Columbus State University, Columbus, GA. This problem was also solved by Christopher York and Luke Meyer, Texas Academy of Leadership in the Humanities, Lamar University, Beaumont, TX, Luke Bent, Alma College, Alma, Michigan, Armstrong Problem Solvers, Armstrong State University, Savannah, GA and Moti Levy, Rehovot, Israel.*

(a) It is well known that any subsequence of a convergent sequence is convergent (to the same limit). If the subsequence of all local extreme terms  $a_{n_k}$ ,  $n_1 < n_2 < \dots$ , is convergent, to  $L$ , then let us show that the sequence  $\{a_n\}$  is convergent to  $L$ . Given and  $\epsilon > 0$ , there exists  $k_0$  such that for every  $k \geq k_0$  we have  $a_{n_k} \in [L - \epsilon, L + \epsilon]$ . For every  $n$ ,  $n_k < n < n_{k+1}$ ,  $\{a_n\}$  must be either non-increasing or non-decreasing otherwise we have another local extreme between  $n_k$  and  $n_{k+1}$  which contradicts the definition of  $n_k$ . This forces  $a_n \in [L - \epsilon, L + \epsilon]$  for every  $n > n_{k_0}$ . Therefore we have  $\lim_{n \rightarrow \infty} a_n = L$ .

(b) Because the new definition requires both inequalities be strict, a term  $a_k$  for which  $a_{k-1} < a_k \leq a_{k+1}$ , it is not considered a local extreme term. As a result we may have a subsequence like this which is has a positive oscillation. A concrete counterexample is to combine the sequence  $(-1)^{\lfloor n/5 \rfloor}$  if  $n \not\equiv 2 \pmod{5}$  with the constant sequence 0 if  $n \equiv 2 \pmod{5}$ , i.e.,

$$1, 1, 0, 1, 1, -1, -1, 0, -1, -1, 1, 1, 0, 1, 1, -1, 1, 0, -1, -1, \dots,$$

in which the subsequence of all local extreme terms is convergent to 0 but the sequence itself is clearly not convergent.

**#1289:** *Proposed by Mike Pinter, Belmont University, Nashville, TN. In honor of the centennial of Pi Mu Epsilon, solve in base 16*

$$\begin{array}{r} PMEMATH \\ + SOCIETY \\ \hline HUNDRED \end{array}$$

(note there are 15 different letters).

*Solution by Jessica Lehr, Elizabethtown College, Elizabethtown, PA 17022. This problem was also solved by Mark Evans, Louisville, KY (email him at markdjevans@twc.com for his code to find all solutions), Thu Dinh, Cal Poly Pomona, CA, Yoshinobru Murayoshi (Okinawa, Japan), Eugen J. Ionascu, Department of Mathematics, Columbus State University, Columbus, GA, Luke Bent, Alma College, Alma, Michigan, Sara Burmeister, North Central College, Naperville, Illinois and Armstrong Problem Solvers, Armstrong State University, Savannah, GA.*

It is clear that  $P + S \leq 16$  because there is no carry from the most significant column. Additionally,  $H + Y = D$  or  $H + Y = 16 + D$ . If  $H + Y = 16 + D$  then  $T + T$  is odd, otherwise if  $H + Y \leq 16$  then  $T + T$  is even. These are a few logical considerations to help lead to a solution.

One possible solution to this problem is:

$$\begin{array}{r} 3 \ 5 \ 13 \ 5 \ 11 \ 6 \ 12 \\ + \ 9 \ 2 \ 1 \ 10 \ 13 \ 6 \ 4 \ . \\ \hline 12 \ 7 \ 15 \ 0 \ 8 \ 13 \ 0 \end{array}$$

Another possible solution would be one in which the value of  $P$  and  $S$  are switched to yield:

$$\begin{array}{r} 9 \ 5 \ 13 \ 5 \ 11 \ 6 \ 12 \\ + \ 3 \ 2 \ 1 \ 10 \ 13 \ 6 \ 4 \ . \\ \hline 12 \ 7 \ 15 \ 0 \ 8 \ 13 \ 0 \end{array}$$

**#1290** *Proposed by Neculai Stanciu, George Emil Palade School, Buzău, Romania and Titu Zvonaru, Comănești, Romania.*

*Consider a set of five distinct positive real numbers such that if we take all products of pairs of these numbers, then only seven distinct numbers are formed. Thus, if the numbers are  $0 < x_1 < x_2 < x_3 < x_4 < x_5$ , if we look at the set formed from all products  $x_i x_j$ , with  $i \neq j$ , then there are only seven distinct numbers. Prove the  $x_i$ 's form a geometric progression; in other words, there is an  $r$  such that  $x_{i+1} = r x_i$  for  $i \in \{1, 2, 3, 4\}$ .*

*Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX 76909 (email contact: charles.diminnie@angelo.edu). This problem was also solved by Mark Evans, Louisville, KY, Thu Dinh, Cal Poly Pomona, CA, Murayoshi Yoshinobu, Okinawa, Japan, David Stoner, South Aiken High School, Henry Ricardo, New York Math Circle, Peter A. Lindstrom, North Lake College, Irving, TX, Eugen J. Ionascu, Department of Mathematics, Columbus State University, Columbus, GA, Christopher York and Luke Meyer, Texas Academy of Leadership in the Humanities, Lamar University, Beaumont, TX, Luke Bent, Alma College, Alma, Michigan, René Sandroni, St. Bonaventure University, St. Bonaventure, NY, Armstrong Problem Solvers, Armstrong State University, Savannah, GA and Moti Levy, Rehovot, Israel.*

Since there are ten possible outcomes for  $x_i x_j$ , with  $i \neq j$ , and only seven of these are distinct, there must be three situations where two such products yield the same answer.

Note first that if, for example,  $x_1x_2 = x_2x_4$ , then  $x_1 = x_4$ , which is impossible. Therefore, in searching for the three situations described above, we must restrict our consideration to the cases where  $x_ix_j = x_mx_n$  for distinct  $i, j, m, n \in \{1, 2, 3, 4, 5\}$ .

Also, if we try  $x_1x_2 = x_3x_4$ , then  $\frac{x_1}{x_3} = \frac{x_4}{x_2}$ , which is impossible since  $\frac{x_1}{x_3} < 1$  while  $\frac{x_4}{x_2} > 1$ . When we eliminate these situations as well, we are left with five possibilities, and exactly three of the five *must* hold (as we have exactly seven distinct numbers):

- (1)  $x_1x_4 = x_2x_3$ .
- (2)  $x_1x_5 = x_2x_3$ .
- (3)  $x_1x_5 = x_2x_4$ .
- (4)  $x_1x_5 = x_3x_4$ .
- (5)  $x_2x_5 = x_3x_4$ .

Clearly, at most one of conditions (2), (3) and (4) can hold. If, for example, (2) and (3) hold, then  $x_2x_3 = x_1x_5 = x_2x_4$  and we get  $x_3 = x_4$ . Similar problems occur when (2) and (4) or (3) and (4) both hold. Therefore, we are down to conditions (1), (5), and exactly one of conditions (2), (3) and (4) hold. However, (1) and (2) imply that  $x_4 = x_5$  while (4) and (5) imply that  $x_1 = x_2$ . Hence, the three conditions that hold must be (1), (3) and (5). These may be re-written as  $\frac{x_2}{x_1} = \frac{x_4}{x_3}$ ,  $\frac{x_2}{x_1} = \frac{x_5}{x_4}$  and  $\frac{x_3}{x_2} = \frac{x_5}{x_4}$ , which combine to give  $\frac{x_2}{x_1} = \frac{x_3}{x_2} = \frac{x_4}{x_3} = \frac{x_5}{x_4}$ . Thus,  $x_1, x_2, x_3, x_4, x_5$  form a geometric progression.

**#1291.** *Proposed by Chirita Marcel, Bucharest, Romania.*

*Given  $x_1, x_2, x_3, x_4, x_5, x_6 \in (0, \infty)$  such that*

$$\frac{1}{x_1 + x_2} + \frac{1}{x_3 + x_4} + \frac{1}{x_5 + x_6} = 1,$$

*prove that*

$$\left( \sum_{i=1}^6 x_i \right)^2 \left( \sum_{i=1}^6 x_i + 9 \right) \geq 54(x_1 + x_2)(x_3 + x_4)(x_5 + x_6).$$

*Solution below by Henry Ricardo, New York Math Circle, henry@mec.cuny.edu. This problem was also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX, by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, by Perfetti Paolo, Dipartimento di Matematica, Università degli studi "Tor Vergata" Roma, Italy, Thu Dinh, Cal Poly Pomona, David Stoner, South Aiken High School, Christopher York, Texas Academy of Leadership in the Humanities, Lamar University, Charles Diminnie and Andrew Siefker, Angelo State University, San Angelo, TX, Eugen J. Ionascu, Department of Mathematics, Columbus State University, Columbus, GA, Corneliu Mănescu-Avram, Transportation High School, Ploiesti, Romania, Armstrong Problem Solvers, Armstrong State University, Savannah, GA and Moti Levy, Rehovot, Israel.*

Let  $a = x_1 + x_2$ ,  $b = x_3 + x_4$ , and  $c = x_5 + x_6$ . Our problem is now equivalent to proving that

$$(a + b + c)^2(a + b + c + 9) \geq 54abc.$$

Noting that  $1/a + 1/b + 1/c = 1$  is equivalent to  $ab + bc + ca = abc$ , we use the arithmetic-harmonic mean inequality to write

$$3 = \frac{3}{1/a + 1/b + 1/c} \leq \frac{a + b + c}{3},$$

so  $a + b + c \geq 9$  and  $a + b + c + 9 \geq 18$ . Therefore it suffices to show that  $(a + b + c)^2 \geq 3abc$ .

Now  $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) = a^2 + b^2 + c^2 + 2abc$ . As the arithmetic-geometric mean inequality implies that  $a^2 + b^2 + c^2 \geq ab + bc + ca = abc$ ,  $(a + b + c)^2 \geq abc + 2abc = 3abc$ , and we are done. We note that equality holds if and only if  $a = b = c = 3$ , or  $x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 3$ .

Alternatively, one could argue as follows. We must show that  $(a + b + c)^3 + 9(a + b + c)^2 \geq 54abc$ . Now the AGM inequality gives us  $a + b + c \geq 3\sqrt[3]{abc}$ , so  $(a + b + c)^3 \geq 27abc$ . Then, as in the previous proof, we can show that  $9(a + b + c)^2 \geq 27abc$ , and we are done.

**#1292.** Proposed by Moti Levy, Rehovot, Israel.

Let  $f(x)$  and  $f^2(x)$  be Riemann-integrable functions on  $[0, 1]$ , and let  $g(x)$  be a twice-differentiable function on  $[0, 1]$  such that  $g(0) = 1$ .

a) Show that

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n g\left(\frac{1}{n} f\left(\frac{k}{n}\right)\right) = \exp\left(g'(0) \int_0^1 f(x) dx\right).$$

b) Find a suitable choice of the functions  $f(x)$  and  $g(x)$  to solve **Problem 1892** from Mathematics Magazine (proposed by Jose Luis Diaz-Barrero):

$$\lim_{n \rightarrow \infty} \frac{1}{n^n} \prod_{k=1}^n \frac{n\sqrt{n} + (n+1)\sqrt{k}}{\sqrt{n} + \sqrt{k}} = \frac{4}{e}.$$

*Solution by Perfetti Paolo, Dipartimento di Matematica, Università degli studi "Tor Vergata" Roma, Italy. This problem was also solved by Christopher York, Texas Academy of Leadership in the Humanities (early entrance program), Lamar University, Beaumont, Texas.*

a) Since  $f$  is integrable, it is bounded:  $\sup_{0 \leq x \leq 1} |f(x)| \leq M$ . It follows that  $\frac{1}{n} |f(\frac{k}{n})| \leq \frac{M}{n}$ . We have  $g(x) = g(0) + g'(0)x + o(x^2)$  and  $\ln(1+x) = x + o(x)$ , so

$$g\left(\frac{k}{n}\right) = g(0) + g'(0)\frac{1}{n}f\left(\frac{k}{n}\right) + o\left(\frac{1}{n}\right) = 1 + g'(0)\frac{1}{n}f\left(\frac{k}{n}\right) + o\left(\frac{1}{n}\right) > 0.$$

It follows that

$$\begin{aligned} \ln \left( \prod_{k=1}^n g \left( \frac{1}{n} f \left( \frac{k}{n} \right) \right) \right) &= \sum_{k=1}^n \ln \left( g \left( \frac{1}{n} f \left( \frac{k}{n} \right) \right) \right) \\ &= \sum_{k=1}^n \ln \left( 1 + g'(0) \frac{1}{n} f \left( \frac{k}{n} \right) + o \left( \frac{1}{n} \right) \right) = \sum_{k=1}^n \left( g'(0) \frac{1}{n} f \left( \frac{k}{n} \right) + o \left( \frac{1}{n} \right) \right) \\ &= g'(0) \sum_{k=1}^n \frac{1}{n} f \left( \frac{k}{n} \right) + n o \left( \frac{1}{n} \right) = g'(0) \sum_{k=1}^n \frac{1}{n} f \left( \frac{k}{n} \right) + o(1). \end{aligned}$$

The limit is clearly the Riemann–sum

$$g'(0) \int_0^1 f(x) dx,$$

whence the limit upon exponentiating is

$$\exp \left( g'(0) \int_0^1 f(x) dx \right).$$

b) Straightforward algebra yields

$$\lim_{n \rightarrow \infty} \frac{1}{n^n} \prod_{k=1}^n \frac{n\sqrt{n} + (n+1)\sqrt{k}}{\sqrt{n} + \sqrt{k}} = \lim_{n \rightarrow \infty} \prod_{k=1}^n \left( 1 + \frac{1}{n} \frac{\sqrt{\frac{k}{n}}}{1 + \sqrt{\frac{k}{n}}} \right)$$

so we have  $g(x) = 1 + x$ ,  $f(x) = \sqrt{x}$  and then

$$\exp \left\{ 1 \cdot \int_0^1 \frac{\sqrt{x}}{1 + \sqrt{x}} dx \right\} = \frac{4}{e}$$

after trivial integrations (for example, let  $x = t^2$  and then replace the resulting  $t^2$  in the numerator with  $((t+1) - 1)^2$ ).

**#1293.** *Proposed by Steven J. Miller, Williams College.*

*The following is from the 2010 Green Chicken Contest between Middlebury and Williams.*

*Every year Middlebury and Williams have a math competition among their students, with the winning team getting to keep the infamous Green Chicken till the following year; see*

[http://web.williams.edu/Mathematics/sjmillers/public\\_html/greenchicken/index.htm](http://web.williams.edu/Mathematics/sjmillers/public_html/greenchicken/index.htm)

*for pictures and additional history and problems. The following is a modification of a problem from 2010.*

*Instead of taking a math contest, Middlebury and Williams decide to settle who gets the Green Chicken by playing the following game. Consider the first one million positive integers. Player A's goal is to choose 10,000 of these numbers such that at the end of the choosing procedure there are at least 20 pairs of chosen integers with the same positive difference (for example, (12,39), (39,66) and (101,128) count as three pairs with a difference of 27). A turn consists of Player A choosing 10 numbers, and then Player B moving up to 10 of any number chosen to any unchosen number. We keep playing until A has chosen 10,000*



numbers, allowing  $B$  to get its final turn. Determine which player has a winning strategy, and prove your claim.

*Solution below by **Mark Evans of Louisville, KY**. This problem was also solved by **Armstrong Problem Solvers, Armstrong State University, Savannah, GA***

On the last turn,  $B$  can remove up to 10 numbers leaving at least 9,990 numbers. From those 9,990 numbers, there will be at least  $9990 \times 9,989/2$  differences. As the minimum difference is 1 and the maximum difference is 999,999, we have 999,999 possible differences, some or all of which can occur multiple times.

The average number of times a given difference occurs is  $(9990 \times 9989/2)/999999$ , which is about 49.9 Thus  $A$  must win regardless of what criteria  $A$  uses for choosing the numbers.  $B$  can delay by rejecting  $A$ 's numbers, but the rules still require the game to continue until  $A$  has chosen 10,000 minus 10 numbers.

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