PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2015

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2015

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.

#1300: *D. M. Bătineu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School Buzău, Romania.*

Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n/n! = a > 0$. Find

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} \right).$$

#1301: *Kenneth B. Davenport, Dallas, PA.*

Earlier problems in this journal concerned determining closed form solutions to sums of pentagonal numbers (a solution is given in volume 12, number 7, Fall 2007, pages 433–434, problem #1147). Consider more generally the sum of reciprocals of polygonal numbers with an odd number of sides; explicitly, prove

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{2}{((2n+1)k-(2n-1))k} \\ &= \frac{\pi}{2n-1}\left(\csc\left(\frac{2\pi}{2n+1}\right) - \tan\left(\frac{\pi}{2n+1}\right)\right) + \frac{2\log(4n+2)}{2n-1} \\ &\quad -\frac{4}{2n-1}\sum_{j=1}^{n}\cos\left(\frac{4j\pi}{2n+1}\right)\log\left(\sin\left(\frac{\pi j}{2n+1}\right)\right). \end{aligned}$$

#1302: *Steven Finch, Harvard University, Cambridge, MA.*

Date: March 19, 2015.

Let A, B, C, D be independent uniform random points on the unit sphere Σ in $\mathbb{R}^3$. The points A, B, C determine a unique disk Δ inscribed within Σ almost surely. Let Γ denote the oblique circular cone with base Δ and apex D. The volume ω of Γ cannot exceed $32\pi/81$. Find the probability density function for ω in closed-form. Find the first and second moments of ω as well.

Note: The density function here is, in fact, algebraic in ω! This is believed to be rare for such problems in geometric probability.

#1303: *E. Ionascu and R. Stephens, Columbus State University, Columbus, GA.*

Suppose we have K dollars in an account that accumulates compound interest at the rate of $i > 0$ per time period and that a payment of P is made from that account at the end of each time period for n periods such that the balance in the account after the last payment is zero. This is an example of what, in Financial Mathematics, is called an *Ordinary Annunity Certain with*

- Effective Interest Rate i,
- Discount Factor $v = \frac{1}{1+i}$, and
- Unit Present Value $a = \frac{K}{P} = v + v^2 + \cdots + v^n = \frac{1-v^n}{i}$.

When $n > 2$, K, and P (and therefore a) are known, it is desirable to determine the interest rate i. Note that if $n = 2$, then the quadratic $a = (1+i)^{-1} + (1+i)^{-2}$ is easily solved for i, but for $n > 2$ it is difficult or impossible to find a closed form exact solution for i in the equation $a = (1+i)^{-1} + \cdots + (1+i)^{-n}$. Various methods for estimating i are well known. For example, $\frac{2(n-a)}{a(n+1)}$ is a good estimate when n is small and $\frac{1-(\frac{a}{n})^2}{a}$ is a good estimate when n is large.

Consider $i_* > 0$ as an estimate for i. Then our estimated Discount Factor is $v_* = \frac{1}{1+i^*}$ and our estimated Unit Present Value is $a_* = v_* + v_*^2 + ... + v_*^n = \frac{1-v_*^n}{i_*}$. Show that

$$i_{**} \;=\; i_* \frac{a_*(\frac{a_*}{a}) - nv_*^{n+1}}{a_* - nv_*^{n+1}}.$$

is positive and a better estimate for i than i_*, in the sense that if $i_* < i$ then $i_{**} > i_*$ and if $i < i_*$ then $i_{**} < i_*$. Note that equation (1) appears in JFEP, V. 13, No. 2. The empirical evidence indicates that this significantly improves any reasonable estimate of i, but it is not the case that i_{**} is always between i and i_*. Establishing the conditions under which $|i - i_{**}| < |i - i_*|$ is an open problem.

#1304: *Steven J. Miller, Williams College, Williamstown, MA.*

The following problem is a generalization of one from the 2014 Green Chicken competition between Middlebury and Williams. We say a positive integer n is **k-ladderful** if we have

$$n \;=\; p_1^1 p_2^2 \cdots p_k^k,$$

where we do not assume the primes are adjacent, distinct, or even in increasing order. We give a few examples.

- The only 0-ladderful number is 1.
- The 1-ladderful numbers are the primes.

- The 2-ladderful numbers are either of the form pq^2 (for two distinct primes) or $pp^2 = p^3$. For example, $75 = 3 \cdot 5^2$, $98 = 2 \cdot 7^2$, $44 = 11 \cdot 2^2$ and $8 = 2 \cdot 2^2 = 2^3$ are all 3-ladderful.
- There are a lot more possibilities for 3-ladderful numbers. Let p, q and r denote three distinct primes. They could be of the form pq^2r^3, or $p^3r^3 = p \cdot p^2 \cdot r^3$, or $pq^5 = p \cdot q^2 \cdot q^3$, or $p^4q^2 = p \cdot q^2 \cdot p^3$, or finally $p^6 = p \cdot p^2 \cdot p^3$.

Let $\mathcal{L}$ denote the set of ladderful numbers; this means $n \in \mathcal{L}$ if and only if there is a k such that n is a k-ladderful number. Determine the growth rate of the ladderful numbers. Explicitly, if $\mathcal{L}(x)$ is the number of ladderful numbers at most x, find constants r and δ such that there

$$0 \ < \ C_1 \ \leq \ \lim_{x\to\infty} \frac{\mathcal{L}(x)}{x^r / \log^\delta x} \ \leq \ C_2 \ < \ \infty.$$

#1305: *Steven J. Miller, Williams College, Williamstown, MA.*

Let $\mathbb{N}_{\rm twin}$ be the set of all integers whose only prime factors are twin primes (we say p is a twin prime if it is prime and either $p+2$ or $p-2$ is also prime, as except for 2 and 3 all neighboring primes are at least 2 units apart). Thus 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21 and 25 are all in $\mathbb{N}_{\rm twin}$ while 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 23 and 24 are not. Does

$$\mathcal{S} \ := \ \sum_{n\in\mathbb{N}_{\rm twin}} \frac{1}{n}$$

converge or diverge? If it converges approximate the sum.; if it diverges approximate (as a function of x) $\mathcal{S}(x) := \sum_{n\in\mathbb{N}_{\rm twin}, n\leq x} 1/n$.

2. Solutions

Problems solved after the last issue went to press: #1290 by the Skidmore College Problem Group.

#1294: *Proposed by Chirita Marcel, Bucharest, Romania. Let f be a differentiable function such that, for some a, b satisfying $0 < a < b < 1$ we have*

$$\int_0^a f(x)dx \ = \ \int_b^1 f(x)dx \ = \ 0.$$

Prove that

$$\left|\int_0^1 f(x)dx\right| \ \leq \ \frac{1-a+b}{4} \sup_{x\in(0,1)} |f'(x)|.$$

Solution below by ***Panagiotis T. Krasopoulos, Social Insurance Institute, Athens, Greece****, solved at the same time by* ***E. Ionascu, Columbus State University, Columbus, GA****. Also solved by* ***Ethan Gegner, Taylor University, Upland, IN****,* ***Perfetti***

Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, C. Edwards, E. Ionascu and R. Stephens, Columbus State University, Columbus, GA, Mark Evans, Louisville, KY, Hongwei Chen, Christopher Newport University, the Missouri State Problem Solving Group.

Let us define the following function, which is twice differentiable in $(0, 1)$,

$$F(x) = \int_0^x f(s)ds.$$

It is also true that $F(0) = F(a) = 0$, $F(b) = F(1) = \int_0^1 f(s)ds$. Next let us use three times the Quadratic Mean Value Theorem. Thus, there are $\xi_1, \xi_2, \xi_3 \in (0, 1)$, such that

$$\begin{aligned} F(0) - F(b) &= F^{'}(b)(0 - b) + F^{''}(\xi_1)\frac{(0 - b)^2}{2} \\ F(a) - F(b) &= F^{'}(b)(a - b) + F^{''}(\xi_2)\frac{(a - b)^2}{2} \\ F(1) - F(b) &= F^{'}(b)(1 - b) + F^{''}(\xi_3)\frac{(1 - b)^2}{2}. \end{aligned}$$

The last gives $F^{'}(b) = F^{''}(\xi_3)\frac{b-1}{2}$. We substitute this into the other two equations above and find

$$\begin{aligned} -F(b) &= F^{''}(\xi_3)\frac{b - b^2}{2} + F^{''}(\xi_1)\frac{b^2}{2} \\ -F(b) &= F^{''}(\xi_3)\frac{(b - 1)(a - b)}{2} + F^{''}(\xi_2)\frac{(a - b)^2}{2}. \end{aligned}$$

Since $F^{''}(x) = f^{'}(x)$, the above equations become

$$|F(b)| \le |F^{''}(\xi_3)|\frac{b - b^2}{2} + |F^{''}(\xi_1)|\frac{b^2}{2} \le \frac{b}{2} \sup_{x\in(0,1)} |f^{'}(x)|,$$

and

$$|F(b)| \le |F^{''}(\xi_3)|\frac{(b - 1)(a - b)}{2} + |F^{''}(\xi_2)|\frac{(a - b)^2}{2} \le \frac{(b - a)(1 - a)}{2} \sup_{x\in(0,1)} |f^{'}(x)|.$$

By adding the above inequalities we get

$$|F(b)| = \left|\int_0^1 f(s)ds\right| \le \frac{b + (b - a)(1 - a)}{4} \sup_{x\in(0,1)} |f^{'}(x)|.$$

The upper bound that we have found is better than the problem's bound. Since $0 < b-a < 1$ we have $b + (b - a)(1 - a) < b + 1 - a$. This completes the proof.

#1295: *Proposed by Moti Levy, Rehovot, Israel.* This problem is related to **Problem 1889** proposed by Gary Gordon and Peter McGrath in *Mathematics Magazine*. For every positive integer k, consider the series

$$S_k = \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}\right)-\left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}\right)$$
$$+\left(\frac{1}{2k+1}+\frac{1}{2k+2}+\cdots+\frac{1}{3k}\right)-\left(\frac{1}{3k+1}+\frac{1}{3k+2}+\cdots+\frac{1}{4k}\right)\pm\cdots.$$

(a) Show that $H_k > S_k > \alpha H_k$ for some α, $0<\alpha<1$. $H_k = \sum_{m=1}^{k}\frac{1}{m}$ is the harmonic series.
(b) Prove that $\lim_{k\to\infty}(H_k - S_k) = \ln\frac{\pi}{2}$.
(c) Find a closed form of S_k (with finite number of summands).
Check your solution by evaluating $S_8 = \left(\frac{1}{16}+\frac{\sqrt{2}}{8}+\frac{1}{4}\sqrt{\sqrt{2}+2}\right)\pi+\frac{1}{8}\ln 2$.
(d) Show that $\lim_{k\to\infty}\left(\sum_{m=1}^{k-1}\frac{\pi}{2k\sin\frac{m\pi}{k}}-\ln\frac{2k}{\pi}\right)=\gamma$, where γ is the Euler-Mascheroni constant.

Solution by ***Hongwei Chen, Department of Mathematics, Christopher Newport University, Newport News, Virginia****. Also solved by* ***E. Ionascu, Columbus State University, Columbus, GA***.

First, we establish two expressions for S_k:

(1) $S_k = \sum_{n=0}^{\infty}(-1)^n(H_{(n+1)k}-H_{nk})$, where $H_0 := 0$.
(2) $S_k = \int_0^1 \frac{1+x+x^2+\cdots+x^{k-1}}{1+x^k}\,dx$.

The expression (1) follows from the definition of the harmonic series directly. The solution to Math Magazine Problem 1889 gave a proof of the expression (2). We offer another proof here. Define

$$f_k(x) = (1+x+\cdots+x^{k-1})-(x^k+x^{k+1}+\cdots+x^{2k-1})$$
$$+(x^{2k}+x^{2k+1}+\cdots+x^{3k-1})-(x^{3k}+x^{3k+1}+\cdots+x^{4k-1})+\cdots.$$

The absolutely convergence on $(-1,1)$ of $f_k(x)$ enables us to rearrange $f_k(x)$ as

$$f_k(x) = (1-x^k+x^{2k}-x^{3k}+\cdots)+(x-x^{k+1}+x^{2k+1}-x^{3k+1}+\cdots+)$$
$$+\cdots+(x^{k-1}-x^{2k-1}+x^{3k-1}-x^{4k-1}+\cdots).$$

Summing these separate geometric series yields

$$f_k(x) = \frac{1}{1+x^k}+\frac{x}{1+x^k}+\cdots+\frac{x^{k-1}}{1+x^k} = \frac{1+x+\cdots+x^{k-1}}{1+x^k}.$$

Now integrating $f_k(x)$ gives

$$S_k = \int_0^1 f_k(x)\,dx = \int_0^1 \frac{1+x+x^2+\cdots+x^{k-1}}{1+x^k}\,dx$$

as desired.

(a). Note that

$$H_k \;=\; \sum_{n=1}^{k} \frac{1}{n} \;=\; \int_0^1 (1 + x + \cdots + x^{k-1})\, dx.$$

Thus, by (2) we have

$$H_k - S_k \;=\; \int_0^1 (1 + x + \cdots + x^{k-1}) \left(1 - \frac{1}{1+x^k}\right) dx \;>\; 0.$$

On the other hand, again by (2) we have

$$S_{n+1} - S_n \;=\; \int_0^1 \frac{2x^n}{(1+x^n)(1+x^{n+1})}\, dx \;>\; \int_0^1 \frac{x^n}{1+x^{n+1}}\, dx \;=\; \frac{1}{n+1} \ln 2,$$

where the inequality $1 + x^n > 2x^n$ in $[0, 1)$ is used. Therefore,

$$S_k \;=\; S_1 + \sum_{n=1}^{k-1} (S_{n+1} - S_n) \;>\; \ln 2 + \left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) \ln 2 \;=\; (\ln 2) H_k.$$

This shows part (a) with $0 < \alpha = \ln 2 < 1$.

(b). Recall that

$$H_k \;=\; \ln k + \gamma + \frac{1}{2k} + O(1/k^2),$$

where γ is the Euler-Mascheroni constant. We find that

$$H_{(n+1)k} - H_{nk} \;=\; \ln\left(\frac{n+1}{n}\right) - \frac{1}{2kn(n+1)} + \frac{1}{k^2}\, O(1/n^2).$$

By (1) we obtain that

$$H_k - S_k \;=\; \sum_{n=1}^{\infty} (-1)^{n+1} \ln\left(\frac{n+1}{n}\right) + \frac{1}{2k} \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} + \frac{1}{k^2}\, O\left(\sum 1/n^2\right),$$

and so

$$\lim_{k\to\infty} (H_k - S_k) \;=\; \sum_{n=1}^{\infty} (-1)^{n+1} \ln\left(\frac{n+1}{n}\right) \;=\; \ln\left(\prod_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{(-1)^{n+1}}\right).$$

This, together with the Wallis formula

$$\prod_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^{(-1)^{n+1}} \;=\; \frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7} \cdots \;=\; \frac{\pi}{2}$$

proves the part (b).

(c). We show that

$$S_k \;=\; \frac{1}{k} \ln 2 + \frac{\pi}{2k} \sum_{l=1}^{k-1} \csc\left(\frac{l\pi}{k}\right). \qquad (*)$$

To this end, first notice that

$$\int_0^1 \frac{x^{k-1}\, dx}{1+x^k} \;=\; \frac{1}{k} \int_0^1 \frac{dt}{1+t} \;=\; \frac{1}{k} \ln 2.$$

We then observe that

$$\int_0^1 \frac{1+x+\cdots+x^{k-2}}{1+x^k}\,dx \;=\; \frac{1}{2}\sum_{l=1}^{k-1}\int_0^1 \frac{x^{l-1}+x^{k-l-1}}{1+x^k}\,dx.$$

The substitution $x = 1/t$ yields

$$\int_0^1 \frac{x^{k-l-1}}{1+x^k}\,dx \;=\; \int_1^\infty \frac{t^{l-1}}{1+t^k}\,dt.$$

Euler's reflection formula implies

$$\int_0^1 \frac{x^{l-1}+x^{k-l-1}}{1+x^k}\,dx \;=\; \int_0^\infty \frac{x^{l-1}}{1+x^k}\,dx \;=\; \frac{\pi}{k}\csc\left(\frac{l\pi}{k}\right).$$

In summary, by (2) we find that

$$S_k \;=\; \int_0^1 \frac{x^{k-1}}{1+x^k}\,dx + \int_0^1 \frac{1+x+\cdots+x^{k-2}}{1+x^k}\,dx \;=\; \frac{1}{k}\ln 2 + \frac{\pi}{2k}\sum_{l=1}^{k-1}\csc\left(\frac{l\pi}{k}\right)$$

as claimed. Moreover, appealing to the facts that

$$\csc\left(\frac{\pi}{2}\right) \;=\; 1,\;\csc\left(\frac{l\pi}{k}\right) \;=\; \csc\left(\frac{(k-l)\pi}{k}\right), \quad (1 \le l < k),$$

we can rewrite

$$S_k \;=\; \begin{cases} \frac{1}{k}\ln 2 + \frac{\pi}{k}\sum_{l=1}^{(k-1)/2}\csc\left(\frac{l\pi}{k}\right), & \text{when } k \text{ is odd,} \\ \frac{1}{k}\ln 2 + \frac{\pi}{2k} + \frac{\pi}{k}\sum_{l=1}^{k/2-1}\csc\left(\frac{l\pi}{k}\right), & \text{when } k \text{ is even.} \end{cases}$$

In particular, we find that

$$\begin{aligned} S_8 \;&=\; \frac{1}{8}\ln 2 + \frac{\pi}{16} + \frac{\pi}{8}\sum_{l=1}^{3}\csc\left(\frac{l\pi}{k}\right) \\ &=\; \frac{1}{8}\ln 2 + \frac{\pi}{16} + \frac{\pi}{8}\left(\csc\left(\frac{\pi}{8}\right) + \csc\left(\frac{\pi}{4}\right) + \csc\left(\frac{3\pi}{8}\right)\right) \\ &=\; \frac{1}{8}\ln 2 + \frac{\pi}{16} + \frac{\pi}{8}\left(\csc\left(\frac{\pi}{4}\right) + 4\cos\left(\frac{\pi}{8}\right)\right) \\ &=\; \frac{1}{8}\ln 2 + \frac{\pi}{16} + \frac{\pi}{8}\left(\sqrt{2} + 2\sqrt{2+\sqrt{2}}\right), \end{aligned}$$

where we have used

$$\csc(\pi/8) + \csc(3\pi/8) \;=\; \frac{\sin(3\pi/8)+\sin(\pi/8)}{\sin(\pi/8)\sin(3\pi/8)}$$

$$= \frac{2\sin(\pi/4)\cos(\pi/8)}{\sin(\pi/8)\cos(\pi/8)} = 4\cos(\pi/8) \;=\; \frac{1}{2}\sqrt{2+\sqrt{2}}.$$

Remark. Using the partial fraction decomposition and the kth roots of -1, the solution of Problem 11499 in *Amer. Math. Monthly* **119**, 2012, p. 254 shows that

$$S_k \;=\; \frac{1}{k}\ln 2 + \frac{\pi}{k^2}\sum_{l=1}^{[k/2]}(k+1-2l)\cot\left(\frac{(2l-1)\pi}{2k}\right).$$

This can be derived by (∗) via the following identity (For example, see H. Chen, Excursions in Classical Analysis, MAA, 2010, P.80)

$$\sum_{l=1}^{k-1} \csc\left(\frac{l\pi}{k}\right) = -\frac{1}{k}\sum_{l=1}^{k}(2l-1)\cot\left(\frac{(2l-1)\pi}{2k}\right).$$

(d). The closed form of S_k in (∗) implies

$$\begin{aligned}\sum_{l=1}^{k-1}\frac{\pi}{2k\sin\left(\frac{l\pi}{k}\right)} - \ln\left(\frac{2k}{\pi}\right) &= S_k - \frac{1}{k}\ln 2 - \ln\left(\frac{2k}{\pi}\right)\\ &= \left(S_k - H_k - \ln\left(\frac{\pi}{2}\right)\right) + H_k - \ln k - \frac{1}{k}\ln 2.\end{aligned}$$

Letting $k \to \infty$, in view of the result (b), we prove that

$$\lim_{k\to\infty}\left(\sum_{l=1}^{k-1}\frac{\pi}{2k\sin\left(\frac{l\pi}{k}\right)} - \ln\left(\frac{2k}{\pi}\right)\right) = \lim_{k\to\infty}(H_k - \ln k) = \gamma.$$

#1296: *Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA 02325*

In this problem all variables represent positive integers. (i) Prove that $a^2 + b^2 = ab$ is impossible. (ii) If a, b, c satisfy $a^2 + b^2 + c^2 = abc$, then prove that $27|abc$. (iii) If a, b, c, d satisfy $a^2 + b^2 + c^2 + d^2 = abcd$, then prove that $16|abcd$. (iv⋆) Prove or disprove: if a, b, c, d, e satisfy $a^2 + b^2 + c^2 + d^2 + e^2 = abcde$, then $9|abcde$. More generally, what might be true for the sum of n squares?

Solution below by ***E. Ionascu at CSU, Columbus, GA***. *Also solved by* ***Josiah Banks, Youngstown State University, Youngstown, Ohio***, ***the Missouri State Problem Solving Group***.

$\boxed{(i)}$ There is a common idea to all parts, which is to treat each equation as a quadratic in one of the variables. In this case, solving for a we get $a = \frac{b\pm\sqrt{b^2-4b^2}}{2} = b\frac{1\pm\sqrt{3}i}{2}$ and so a/b which is a positive rational cannot be at the same time equal to a pure complex number.

$\boxed{(ii)}$ Again, solving for a we get $a = \frac{bc\pm\sqrt{\Delta}}{2}$ where $\Delta := b^2c^2 - 4(b^2 + c^2)$. If b and c are not divisible by 3, then $a^2 \equiv b^2 \equiv 1 \pmod 3$. Hence $\Delta := b^2c^2 - 4(b^2 + c^2) \equiv 1 - 4(2) = -7 \equiv 2 \pmod{3}$. Hence Δ cannot be a perfect square. It follows that b or c is divisible by 3. If only one of them is divisible by 3 then we get $\Delta \equiv -4 \equiv 2 \pmod 3$ and again Δ cannot be a perfect square. It follows that b and c are both divisible by 3 and then a must be divisible by 3. Hence $27|abc$. We include several solutions of this Diophantine equation:

$$(a, b, c) \in \mathcal{S} := \{(3, 3, 3), (3, 3, 6), (3, 6, 15), (3, 15, 39), \cdots\}.$$

Remark: It seems that these solutions are generated in the following way: first $b = c = 3$ gives $a = \frac{bc\pm\sqrt{\Delta}}{2} = \frac{9\pm3}{2}$. So , we obtain two solutions for a, one that we know, $a = 3$, and a new one: $a = 6$. These are essentially the first two solutions in $\mathcal{S}$. Next, we can take $b = 3$ and $c = 6$ in the quadratic formula, and obtain ($\Delta = 9[6^2 - 4(1 + 4)] = 9(16)$) one solutions that we already know $a = 3$ and a new one: $a = (1/2)[3(6) \pm 3(4)] = 15$. So, we have essentially the third solution in $\mathcal{S}$. This proceeders can continue indefinitely and it suggests that we have infinitely many solutions.

$\boxed{(iii)}$ In this case $a = \frac{bcd\pm\sqrt{\Delta}}{2}$ where $\Delta := b^2c^2d^2 - 4(b^2+c^2+d^2)$. If all b, c and d are odd then we know that $b^2 = (2k+1)^2 = 4k(k+1) + 1 \equiv 1 \pmod 8$. Hence, $\Delta \equiv 1 - 4(3) = -11 \equiv 5 \pmod 8$. Because a perfect square is only congruent with 0, 1, or 4 (mod 8) we cannot have Δ a square. So, at least one of the numbers b, c and d is even. Since the whole problem is symmetric in a, b, c and d we may assume that another of these numbers is even. If only two are even, say a and b, then the given equation implies $c^2 + d^2 \equiv 2 \equiv abcd \equiv 0 \pmod 4$ which is not possible. If only three of the numbers are even, again we obtain a similar contradiction. It remains that all the numbers are even, which proves that $16|abcd$. We include some of the solutions of this Diophantine equation

$$(a/2, b/2, c/2, d/2) \in \mathcal{S} \ := \ \{(1,1,1,1),\ (1,1,1,3),\ (1,1,3,11),$$

$$(1,1,11,41),\ (1,1,41,153),\ (1,3,11,131), \cdots\}.$$

$\boxed{(iv\star)}$ We are going to show that the claim $9|abcde$ is true. As before, we have $a = \frac{bcde\pm\sqrt{\Delta}}{2}$ where $\Delta := b^2c^2d^2e^2 - 4(b^2 + c^2 + d^2 + e^2)$. If all of the numbers involved are not divisible by 3, we have $\Delta \equiv 1 - 4(4) = -15 \equiv 0 \pmod 3$. In order for Δ to be a perfect square it is necessary that we must have $9|\Delta$ since $3|\Delta$. Surprisingly we have the following lemma.
Lemma: For every x, y, z and t in $\{1, 4, 7\}$ then

$$\Delta' := xyzt - 4(x + y + z + t) \equiv 3 \ (mod\ 9).$$

Proof of Lemma: We have to check basically $3^4 = 81$ cases but because symmetry we only have to look at just a few. First, since one of the values in $\{1, 4, 7\}$ has to repeat, if it repeats exactly once but the other two are distinct, we obtain three situations:

$$\begin{aligned}
\Delta' &= (1)(1)(4)(7) - 4(1 + 1 + 4 + 7) = -24 \equiv 3 \ (mod\ 9) \\
\Delta' &= (1)(4)(4)(7) - 4(1 + 4 + 4 + 7) = 45 \equiv 3 \ (mod\ 9) \\
\Delta' &= (1)(4)(7)(7) - 4(1 + 4 + 7 + 7) = 120 \equiv 3 \ (mod\ 9).
\end{aligned}$$

If one repeats once and the other two are equal we get three more cases:

$$\begin{aligned}
\Delta' &= (1)(1)(4)(4) - 4(1 + 1 + 4 + 4) = -24 \equiv 3 \ (mod\ 9) \\
\Delta' &= (1)(1)(7)(7) - 4(1 + 4 + 4 + 7) = -15 \equiv 3 \ (mod\ 9) \\
\Delta' &= (4)(4)(7)(7) - 4(1 + 4 + 7 + 7) = 696 \equiv 3 \ (mod\ 9).
\end{aligned}$$

If it repeats exactly twice, we have six possibilities:

$$\begin{aligned}
\Delta' &= (1)(1)(1)(7) - 4(1+1+1+7) = -33 \equiv 3 \ (mod\ 9) \\
\Delta' &= (1)(1)(1)(4) - 4(1+1+1+4) = -24 \equiv 3 \ (mod\ 9) \\
\Delta' &= (1)(4)(4)(4) - 4(1+4+4+4) = 12 \equiv 3 \ (mod\ 9) \\
\Delta' &= (7)(4)(4)(4) - 4(7+4+4+4) = 372 \equiv 3 \ (mod\ 9) \\
\Delta' &= (1)(7)(7)(7) - 4(1+7+7+7) = 255 \equiv 3 \ (mod\ 9) \\
\Delta' &= (4)(7)(7)(7) - 4(4+7+7+7) = 1272 \equiv 3 \ (mod\ 9).
\end{aligned}$$

Finally, if it repeats four times we have only three possibilities:

$$\begin{aligned}
\Delta' &= (1)(1)(1)(1) - 4(1+1+1+1) = -15 \equiv 3 \ (mod\ 9) \\
\Delta' &= (4)(4)(4)(4) - 4(4+4+4+4) = 192 \equiv 3 \ (mod\ 9) \\
\Delta' &= (7)(7)(7)(7) - 4(7+7+7+7) = 2289 \equiv 3 \ (mod\ 9). \quad \square
\end{aligned}$$

Since the remainders modulo 9 of b^2, c^2, d^2 and e^2 can only be in $\{1, 4, 7\}$, our lemma implies that Δ cannot be a perfect square unless at least one of the numbers b, c, d or e is a multiple of 3. In this case Δ must be a perfect square divisible by 9 (and this is indeed possible $a = 3$ $b = 3$, $c = 1$, $d = 1$, and $e = 4$), and so $a = \frac{bcde \pm \sqrt{\Delta}}{2}$ is also divisible by 3. This is enough to conclude that $9|abcde$, completing the proof.

We include some of the solutions of this Diophantine equation

$$(a, b, c, d, e) \in \mathcal{S} := \{(1, 1, 3, 3, 4),\ (1, 1, 3, 3, 5),\ (1, 1, 3, 4, 9),$$

$$(1, 1, 3, 5, 12),\ (1, 1, 3, 9, 23),\ (1, 1, 3, 12, 31), \cdots\}.$$

$\boxed{(v\star)}$ In this case, the equation becomes $\sum_{i=1}^{6} x_i^2 = \prod_{i=1}^{6} x_i$. We see that if there is any solution, x_i cannot be all odd, since the left hand side turns out to be even and the right hand side odd. Also, if exactly one number is even then the left hand side becomes odd and the right side even. So, we need to have at least two of the numbers even. So, the equation can be reduced to

$$4x_1'^2 + 4x_2'^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 4x_1'x_2'x_3x_4x_5x_6.$$

We can do an analysis modulo 3 and conclude that at least three of the numbers must be divisible by 3. So, we could say that $108| \prod_{i=1}^{6} x_i$, but there seems to be no solution of this equation so we do not want to make claims about the empty set, just in case. It is perhaps not that difficult, employing similar techniques, to show that there are no solutions.

#1297: *Proposed by Ben Klein, Davidson College, Davidson, NC.*

Suppose that $p(z)$ is a cubic polynomial over the complex numbers with distinct roots, a_1, a_2, a_3.

(1) *Assuming $p'(a_1) = p'(a_3)$, express a_2 as a function of a_1 and a_3, and express $p'(a_2)$ as a function of $p'(a_1)$.*
(2) *Assuming $p'(a_1) \neq p'(a_3)$, express $p'(a_2)$ as a function of $p'(a_1)$ and $p'(a_3)$.*

Solution below by ***Ethan Gegner, Taylor University, Upland, IN****. Also solved by* ***Panagiotis T. Krasopoulos Social Insurance Institute, Athens, Greece****,* ***Madhav Sharma, Department of Mathematical Sciences, Florida Atlantic University****,*

Eugen J. Ionascu, Department of Mathematics, Columbus State University, Columbus, GA, ***Mark Evans, Louisville, KY***, ***Missouri State University Problem Solving Group, Department of Mathematics, Missouri State University***.

Write $p(z) = C(z - a_1)(z - a_2)(z - a_3)$, so that

$$p'(z) \;=\; C\left(3z^2 - 2(a_1 + a_2 + a_3)z + (a_1a_2 + a_1a_3 + a_2a_3)\right). \tag{2.1}$$

(i) Let $A := p'(a_1) = p'(a_3)$. Then $p'(z) - A = K(z - a_1)(z - a_3)$ for some $K \in \mathbb{C}$, or

$$p'(z) \;=\; Kz^2 - K(a_1 + a_3)z + (a_1a_3 + A). \tag{2.2}$$

Equating coefficients in (2.1) and (2.2) yields $K = 3C$ and $a_2 = \frac{1}{2}(a_1 + a_3)$.

(ii) Let $\alpha = (a_1 - a_2), \beta = (a_1 - a_3), \gamma = (a_2 - a_3)$. Then we have

$$\begin{aligned} p'(a_1) &= C\alpha\beta = C\alpha(\alpha + \gamma) \\ p'(a_3) &= C\beta\gamma = C\gamma(\alpha + \gamma) \\ p'(a_2) &= -C\alpha\gamma \end{aligned}$$

If $p'(a_1) \neq -p'(a_3)$, then it follows that

$$p'(a_2) \;=\; -\frac{p'(a_1)p'(a_3)}{p'(a_1) + p'(a_3)},$$

completing the proof.

#1299: *Proposed by Steven J. Miller, Williams College, Williamstown, MA.*

The following is from the 2014 Green Chicken math competition between Middlebury and Williams Colleges. Consider all sets of 2014 distinct positive integers. For each set, look at all the products of four distinct elements. (a) What is the largest number of distinct products? (b) What is the fewest number of distinct products? Prove your claims.

Solution below by ***Khanh Le, Ohio Wesleyan University, Delaware, Ohio***. *Also solved by* ***Kathleen Lewis, University of the Gambia***, ***Abhay Malik and Thomas Goebeler, The Episcopal Academy, Newtown Square, PA***.

(a) Consider a set of 2014 distinct prime numbers and all of its products of four elements. Any product is clearly unique because it has a distinct prime factorization compared to other products of four. Therefore, there are $\binom{2014}{4}$ distinct products of four elements in the set of 2014 distinct prime numbers, which is the maximum number of products of four possible. Thus, the largest number of distinct products of four in all sets of 2014 distinct positive integers is $\binom{2014}{4}$, or 683,489,813,501.

(b) We first show that there are always at least 8041 distinct products of four elements, and then give a set with exactly that number. We order the elements

$$a_1 \;<\; a_2 \;<\; \cdots \;<\; a_{2013} \;<\; a_{2014}.$$

We show that there are at least 8041 distinct products of four. Indeed, consider the following sequences of products. The first sequence is

$$a_1a_2a_3a_4 \;<\; a_1a_2a_3a_5 \;<\; a_1a_2a_3a_6 \;<\; \cdots \;<\; a_1a_2a_3a_{2014},$$

which has 2011 terms. The second sequence is

$$a_1a_2a_4a_{2014} < a_1a_2a_5a_{2014} < a_1a_2a_6a_{2014} < \cdots < a_1a_2a_{2013}a_{2014},$$

which has 2010 terms. The third sequence is

$$a_1a_3a_{2013}a_{2014} < a_1a_4a_{2013}a_{2014} < a_1a_5a_{2013}a_{2014} < \cdots < a_1a_{2012}a_{2013}a_{2014}$$

which has 2010 terms. We continue this process, with the last sequence being

$$a_2a_{2012}a_{2013}a_{2014} < a_3a_{2012}a_{2013}a_{2014} < a_4a_{2012}a_{2013}a_{2014} < \cdots < a_{2011}a_{2012}a_{2013}a_{2014},$$

which has 2010 terms. By comparing the last term of the first sequence to the first term of the second sequence, it is obvious that every term in the first sequence is strictly smaller than every term in the second sequence. Similarly, the each term in the second sequence is strictly smaller than every term in the third, which is strictly smaller than the fourth and so on. Hence, we have $2011 + 2010 + 2010 + 2010 = 8041$ distinct products. Therefore, we always have at least 8041 distinct products by looking at these products.

Now, we find a set with exactly 8041 distinct products. Consider

$$A = \{2^1, 2^2, 2^3 \cdots 2^{2014}\}$$

(note 2 is arbitrary, and we could have chosen another number). The products of four distinct elements of this set have the form 2^k where k is the sum of four distinct numbers between 1 and 2014. The smallest value of k is $1 + 2 + 3 + 4 = 10$, and the largest one is $2011 + 2012 + 2013 + 2014 = 8050$. We prove that k can be any number between 10 and 8050 by choosing certain sums of four distinct numbers between 1 and 2014 that can cover all numbers from 10 to 8050.

- k can take on values between 10 and 2020 because we have

$$1+2+3+4 = 10, \quad 1+2+3+5 = 11, \quad 1+2+3+6 = 12, \quad \ldots, \quad 1+2+3+2014 = 2020.$$

- k can also be between 2021 and 4030 because we have

$$1+2+4+2014 = 2021, 1+2+5+2014 = 2022, 1+2+6+2014 = 2023, \quad \ldots, \quad 1+2+2013+2014 = 4030$$

- For between 4031 and 6040, we have

$$1+3+2013+2014 = 4031, 1+4+2013+2014 = 4032, \quad \ldots, \quad 1+2012+2013+2014 = 6040$$

- Also between 6041 and 8050, we have

$$2+2012+2013+2014 = 6041, 3+2012+2013+2014 = 6042, \quad \ldots, \quad 2011+2012+2013+2014 = 8050$$

Thus the number of distinct products in this set is $8050 - 10 + 1 = 8041$. Therefore, the fewest number of distinct products is 8041.

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