PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2015

STEVEN J. MILLER (EDITOR)

1. Problems: Fall 2015

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.

The following note from Khanh Le, a student at Ohio Wesleyan, to the editor of the Problem Section beautifully illustrates what we hope people will get out of these pages.

> Last spring semester, I joined the Pi Mu Epsilon Society and first read the math journal PME. I was particularly interested in the problem section. Solving math problems was a large part of my high school math experience that I have forgotten due to the different emphasis in math education in high school and college. However, working and solving problem from the section reminded me of how much I enjoyed doing those little puzzles. I also enjoyed the correspondence with you in which you pointed out how I may have misread the problem, and I kept pushing myself to think more to understand and solve the problem.
>
> I thought it would be more enjoyable if I could do it with my friends. Therefore I decided to start a problem solving club at my school. I was really surprised by how supportive the professors at my school are with the club. The club had participation from both students and professors. Over the course of last semester, we worked on 30 problems from different sources (online, Putnam problems and other competition, and even some basic chess endgames). Most of the problems were collected and proposed by club members. We were not able to solve all of them, but it was definitely a wonderful experience. The section has re-kindled a forgotten interest and inspired me in a wonderful way.

Date: December 1, 2015.

I want to thank you and the problem proposers for all the work you do. Hope that the section will keep inspiring students.

#1306: *Proposed by David Vella, Mathematics and Computer Science Deptartment, Skidmore College, Saratoga Springs, NY 12866.*

Find all integer solutions (p, q) to the equation

$$q^{p+q} + p^p(p+q)^p \;=\; (p^2+q)^q,$$

where p and q are prime numbers.

#1307: *Proposed by Panagiotis T. Krasopoulos, Social Insurance Institute, Athens, Greece.*

Let $p(z)$ be a polynomial with complex coefficients of degree $n \geq 2$ with distinct roots $\alpha_1, \ldots, \alpha_n$ and let $p^{'}(z)$ be its derivative. Prove elementarily (i.e., do not use contour integration and complex analysis) that

$$\sum_{k=1}^{n} \frac{1}{p^{'}(\alpha_k)} \;=\; 0.$$

#1308: *Proposed by Taimur Khalid, Coral Academy of Science LV.*

Consider a triangle ABC. Let the external angle bisectors of angles A and B intersect at a point D, B and C at E, and A and C at F. See Figure 1.

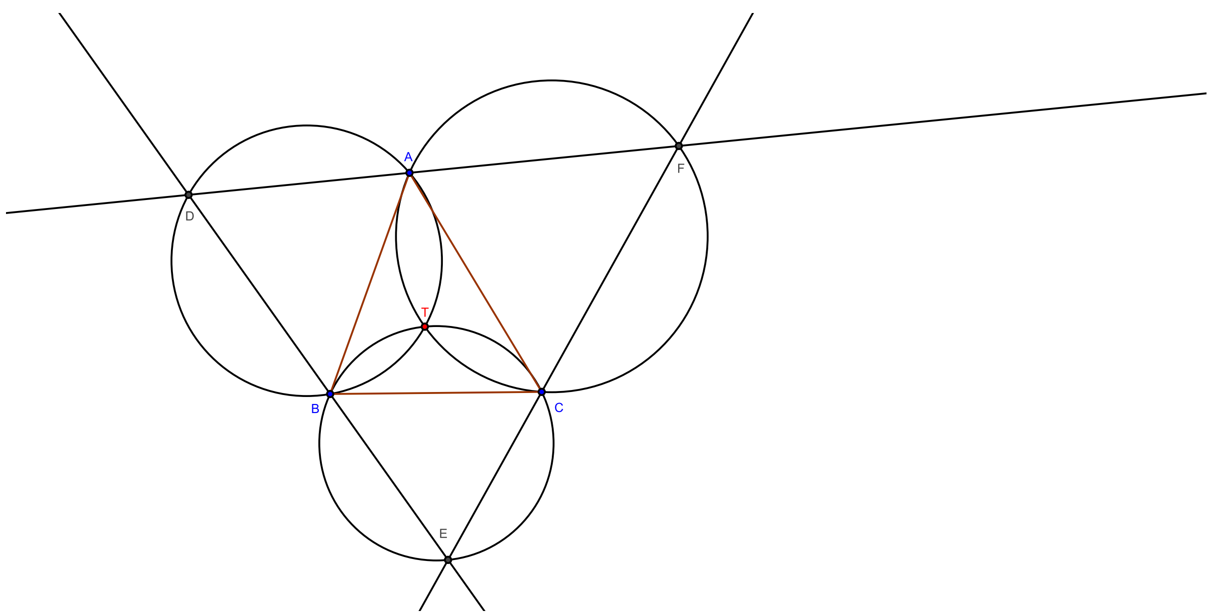

FIGURE 1. Triangle ABC and its external angle bisectors.

(1) Prove that the circumcircles of triangles ADB, BEC, and CFA intersect at a common point.
(2) Prove that this point is the incenter of $\triangle ABC$.

#1309: *Proposed by Kenneth B. Davenport, Dallas, PA.*

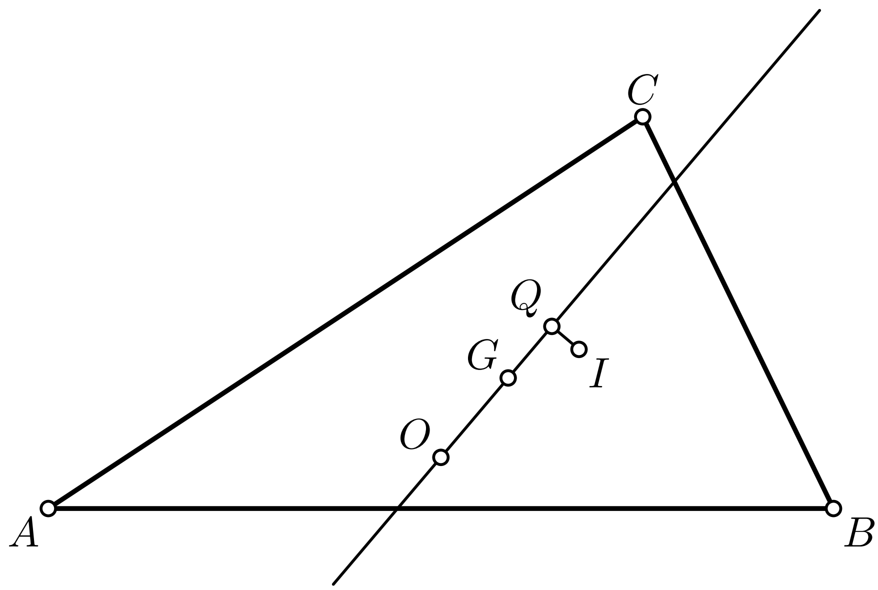

FIGURE 2. A scalene triangle: G is the centroid, O is the circumcenter, I is the incenter, the line is the Euler line GO, Q is the projection of I on the line GO. The length of the segment IQ is the distance from I to the line GO.

The Chebyshev polynomials are defined recursively by $T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x)$ for $N \geq 1$, with $T_0(x) = 1$, $T_1(x) = x$ (and thus $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$). They have many applications in mathematics, especially in approximation theory and polynomial interpolation. As they are of the form $T_N(x) = \cos(N \arccos(x))$, it is interesting to look at cosines (and hence also sines) of arccosines of angles. Prove

$$(-1)^N \cos(N\theta) \;=\; \cos(2N\psi), \quad (-1)^{N+1} \sin(N\theta) \;=\; \sin(2N\psi),$$

where

$$\theta \;=\; \arccos\left(\frac{x}{\sqrt{x^2+4}}\right), \quad \psi \;=\; \arctan\left(\frac{x+\sqrt{x^2+4}}{2}\right).$$

#1310: *Proposed by Sava Grozdev (sava.grozdev@gmail.com), VUZF University, Sofia 1618, Bulgaria and Deko Dekov (ddekov@ddekov.eu), Zahari Knjazheski 81, Stara Zagora 6000, Bulgaria.* This problem is discovered by the computer program "Discoverer" created by Grozdev and Dekov.

Given a scalene triangle ABC with side lengths $a = BC, b = CA$ and $c = AB$. Recall that the *centroid* is the intersection point of the medians of the triangle, the *incenter* is the center of the circle, inscribed in the triangle, and the *circumcenter* is the center of the circle circumcribed around the triangle. Let d be the distance from the incenter of $\triangle ABC$ to the line defined by the centroid of $\triangle ABC$ and the circumcenter of $\triangle ABC$. Find d as a function of a, b and c, that is, $d = f(a, b, c)$; see Figure 2 for an illustration of the problem.

#1311: *Proposed by Abdilkadir Altıntaş, Emirdağ, Afyon, Turkey.*

Compute the product

$$\left(\frac{1}{\sqrt{3}} + \tan 59^\circ\right)\left(\frac{1}{\sqrt{3}} + \tan 58^\circ\right)\cdots\left(\frac{1}{\sqrt{3}} + \tan 2^\circ\right)\left(\frac{1}{\sqrt{3}} + \tan 1^\circ\right).$$

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