PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2016

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2016

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.

#1312: *Proposed by Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA and Stan Wagon, Department of Mathematics, Statistics, and Computer Science, Macalester College, St. Paul, Minnesota.*

Larry Bird and Magic Johnson are playing a game of basketball; they alternating shooting with Bird going first, and the first to make a basket wins. Assume Bird always makes a basket with probability p_B and Magic with probability p_M, where p_B and p_M are independent uniform random variables. (This means the probability each of them is in $[a,b] \subset [0,1]$ is $b-a$, and knowledge of the value of p_B gives no information on the value of p_M.)

(1) What is the probability Bird wins the game?
(2) What is the probability that, when they play, Bird has as good or greater chance of winning than Magic?

Note: for another related problem, see Math Horizons 23:2 (Nov 2015), 30.

#1313: *Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.*

Suppose that $ab+bc+ca = 8abc$ and $a,b,c \geq 1/5$. Prove that $a^2+b^2+c^2+15abc > 15a^2bc+15ab^2c+15abc^2$. Furthermore prove that for any positive constant ϵ the inequality $a^2+b^2+c^2+15abc > 15a^2bc+15ab^2c+15abc^2+\epsilon\, a^2b^2c^2$ does not hold for all $a,b,c \geq 1/5$.

Date: January 8, 2016.

#1314: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*

The following problem is an expanded version of a problem from the 2013 Green Chicken Math Competition between Middlebury and Williams College. Show that there is a positive Fibonacci number that is divisible by 1000, and find the smallest such number; more generally, find all positive integers N such that there exists a positive Fibonacci number F_n which is divisibly by N, and for such N give a bound for the smallest index n that works in terms of N.

#1315: *Proposed by Steven J. Miller, Williams College, and Daniel F. Stone, Bowdoin College.*

Let

$$b_n \;=\; a\left(b + \frac{1}{n}\sum_{i=1}^{n-1} b_i\right) \text{ for } n > 1, \text{ with } b_1 \;=\; ab$$

where $b, a > 0$.

(1) Show that if $a = 1$, then $b_n = H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$ is the n^{th} harmonic number, and thus diverges. Determine the rate of growth of b_n as a function of n.

(2) Show that if $a < 1$, then b_n is bounded and determine the best bound possible.

This problem is motivated by research of the second proposer on political polarization, who proposes a simple mathematical model of affective polarization in which two opposing partisans Bayesian update their beliefs about the opposition's pro-sociality, i.e., degree to which she is willing to sacrifice a personal interest for the greater good. The paper shows that two seemingly unrelated cognitive biases can cause highly distorted, polarized, beliefs about this characteristic for the opposition. One of these biases is called the false consensus bias, which means under-estimation of the extent to which different people have different tastes. Think about how when you're in the mood for ice cream, it's hard to imagine someone wanting a salad. But some people genuinely do prefer salad then, and indeed, you are probably under-estimating the chance of this. Stone shows this misperception can cause actions that are consistent with tastes about what is best for the greater good to appear to be self-serving and thus make one less liked by the opposition, causing affective polarization. He shows a lower-bound for the expectation of such polarization after t periods of interaction is our b_n, in which b is the initial bias (under-estimation of difference in mean tastes for the two parties) and a represents the degree to which new information affects beliefs (this is a function of other parameters of the model).

#1316: *Proposed by Mehtaab Sawhney, Commack High School.*

Consider an $n \times n$ chessboard for $n \geq 2$. Define a left-rook to be a rook that can only attack the squares in the same row to its left. Similarly define right-rooks, up-rooks, and down-rooks. Find the maximum total of right-rooks, left-rooks, up-rooks, and down-rooks, as a function of n, such that no rook is attacking another.

#1317: *Robert C. Gebhardt, Chester, NJ.*

(a) A continuous differentiable function $g(x) > 0$, $a < x < b$, is revolved about the x-axis to create a surface of revolution with area S, and a volume of revolution V. Find all functions $g(x)$, other than $g(x) = 0$ and $g(x) = 2$, such that the surface area S (in square units) is the same number as the volume V (in cubic units) for all finite choices of a and b (the areas of the discs at each end of the volume are not included).

(b) Let $f(x)$ and $g(x)$ be continuous differentiable functions, $a < x < b$, where $0 < f(x) < g(x)$. Each is revolved about the x-axis to create a volume between them. Find all such functions $f(x)$ and $g(x)$ such that the surfaces' total area S (in square units) and the volume V (in cubic units) are the same number. The washers at each end of the volume are not included.

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