PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2015

STEVEN J. MILLER (EDITOR)

1. Problems: Fall 2015

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.

The following note from Khanh Le, a student at Ohio Wesleyan, to the editor of the Problem Section beautifully illustrates what we hope people will get out of these pages.

> Last spring semester, I joined the Pi Mu Epsilon Society and first read the math journal PME. I was particularly interested in the problem section. Solving math problems was a large part of my high school math experience that I have forgotten due to the different emphasis in math education in high school and college. However, working and solving problem from the section reminded me of how much I enjoyed doing those little puzzles. I also enjoyed the correspondence with you in which you pointed out how I may have misread the problem, and I kept pushing myself to think more to understand and solve the problem.
>
> I thought it would be more enjoyable if I could do it with my friends. Therefore I decided to start a problem solving club at my school. I was really surprised by how supportive the professors at my school are with the club. The club had participation from both students and professors. Over the course of last semester, we worked on 30 problems from different sources (online, Putnam problems and other competition, and even some basic chess endgames). Most of the problems were collected and proposed by club members. We were not able to solve all of them, but it was definitely a wonderful experience. The section has re-kindled a forgotten interest and inspired me in a wonderful way.

Date: October 19, 2015.

I want to thank you and the problem proposers for all the work you do. Hope that the section will keep inspiring students.

#1306: *Proposed by David Vella, Mathematics and Computer Science Deptartment, Skidmore College, Saratoga Springs, NY 12866.*

Find all integer solutions (p, q) to the equation

$$q^{p+q} + p^p(p+q)^p \;=\; (p^2+q)^q,$$

where p and q are prime numbers.

#1307: *Proposed by Panagiotis T. Krasopoulos, Social Insurance Institute, Athens, Greece.*

Let $p(z)$ be a polynomial with complex coefficients of degree $n \geq 2$ with distinct roots $\alpha_1, \ldots, \alpha_n$ and let $p^{'}(z)$ be its derivative. Prove elementarily (i.e., do not use contour integration and complex analysis) that

$$\sum_{k=1}^{n} \frac{1}{p^{'}(\alpha_k)} \;=\; 0.$$

#1308: *Proposed by Taimur Khalid, Coral Academy of Science LV.*

Consider a triangle ABC. Let the external angle bisectors of angles A and B intersect at a point D, B and C at E, and A and C at F. See Figure 1.

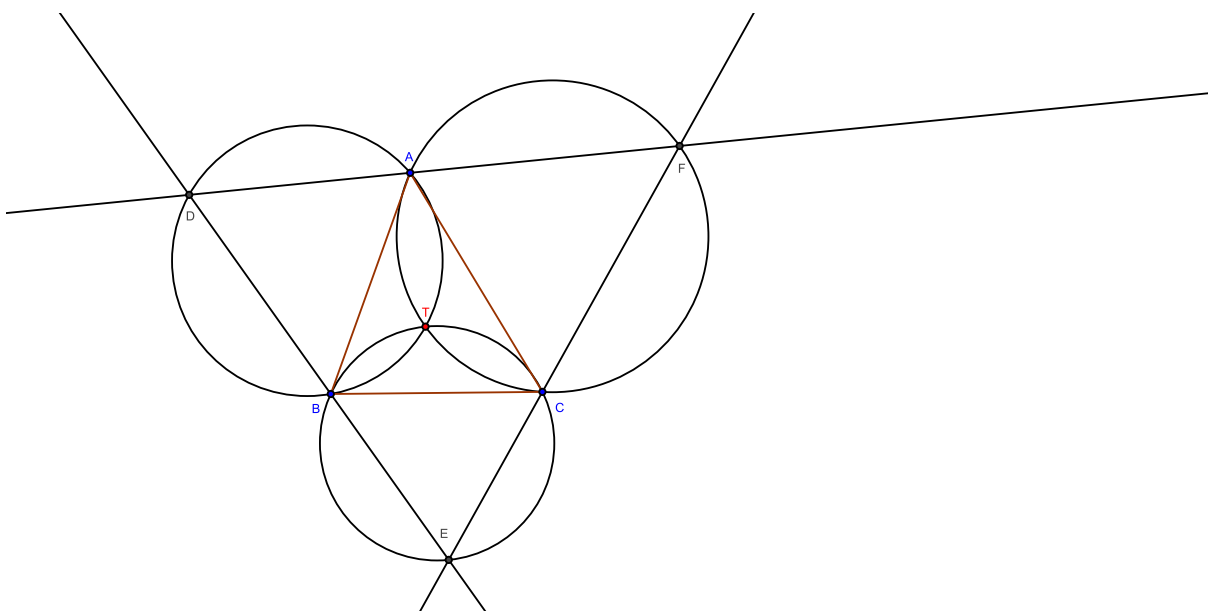

FIGURE 1. Triangle ABC and its external angle bisectors.

(1) Prove that the circumcircles of triangles ADB, BEC, and CFA intersect at a common point.
(2) Prove that this point is the incenter of $\triangle ABC$.

#1309: *Proposed by Kenneth B. Davenport, Dallas, PA.*

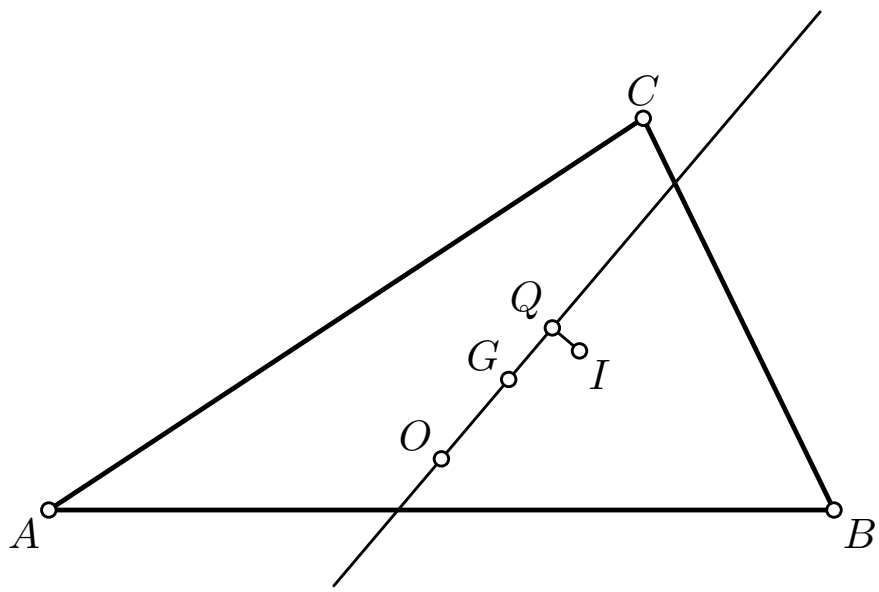

FIGURE 2. A scalene triangle: G is the centroid, O is the circumcenter, I is the incenter, the line is the Euler line GO, Q is the projection of I on the line GO. The length of the segment IQ is the distance from I to the line GO.

The Chebyshev polynomials are defined recursively by $T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x)$ for $N \geq 1$, with $T_0(x) = 1$, $T_1(x) = x$ (and thus $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$). They have many applications in mathematics, especially in approximation theory and polynomial interpolation. As they are of the form $T_N(x) = \cos(N \arccos(x))$, it is interesting to look at cosines (and hence also sines) of arccosines of angles. Prove

$$(-1)^N \cos(N\theta) \;=\; \cos(2N\psi), \quad (-1)^{N+1} \sin(N\theta) \;=\; \sin(2N\psi),$$

where

$$\theta \;=\; \arccos\left(\frac{x}{\sqrt{x^2+4}}\right), \quad \psi \;=\; \arctan\left(\frac{x+\sqrt{x^2+4}}{2}\right).$$

#1310: *Proposed by Sava Grozdev (sava.grozdev@gmail.com), VUZF University, Sofia 1618, Bulgaria and Deko Dekov (ddekov@ddekov.eu), Zahari Knjazheski 81, Stara Zagora 6000, Bulgaria.* This problem is discovered by the computer program "Discoverer" created by Grozdev and Dekov.

Given a scalene triangle ABC with side lengths $a = BC, b = CA$ and $c = AB$. Recall that the *centroid* is the intersection point of the medians of the triangle, the *incenter* is the center of the circle, inscribed in the triangle, and the *circumcenter* is the center of the circle circumcribed around the triangle. Let d be the distance from the incenter of $\triangle ABC$ to the line defined by the centroid of $\triangle ABC$ and the circumcenter of $\triangle ABC$. Find d as a function of a, b and c, that is, $d = f(a, b, c)$; see Figure 2 for an illustration of the problem.

#1311: *Proposed by Abdilkadir Altıntaş, Emirdağ, Afyon, Turkey.*

Compute the product

$$\left(\frac{1}{\sqrt{3}}+\tan 59^\circ\right)\left(\frac{1}{\sqrt{3}}+\tan 58^\circ\right)\cdots\left(\frac{1}{\sqrt{3}}+\tan 2^\circ\right)\left(\frac{1}{\sqrt{3}}+\tan 1^\circ\right).$$

2. Solutions

#1300: *D. M. Bătineu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School Buzău, Romania. Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n/n! = a > 0$. Find*

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right).$$

Solution below by ***Henry Ricardo, New York Math Circle.*** *Also solved by* ***Tommy Goebeler, The Episcopal Academy, Newtown Square, PA***, ***Ángel Plaza, Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, España***, ***Hongwei Chen Department of Mathematics, Christopher Newport University, Newport News, VA***, ***Ethan Gegner,Taylor University, Upland, IN***, *and* ***the Missouri State University Problem Solving Group, Department of Mathematics, Missouri State University***.

The limit equals $1/e$. To see this, we write

$$\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} = \frac{\sqrt[n]{a_n}}{n}\cdot\frac{\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}-1\right)}{\ln\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)}\cdot\ln\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n.$$

Letting $\alpha_n = \sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n}$, we prove the following results which, when combined, yield the desired result:

$$(1)\ \lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}=\frac{1}{e};\quad (2)\ \lim_{n\to\infty}\alpha_n = 1 = \lim_{n\to\infty}\frac{\alpha_n-1}{\ln\alpha_n};\quad (3)\ \lim_{n\to\infty}\alpha_n^n = e.$$

Proof of (1): Let $x_n = a_n/n^n$. Noting that

$$\frac{a_{n+1}}{a_n} = (n+1)\cdot\frac{\frac{a_{n+1}}{(n+1)!}}{\frac{a_n}{n!}},$$

we find that

$$\frac{x_{n+1}}{x_n} = \frac{a_{n+1}}{a_n}\cdot\frac{n^n}{(n+1)^{n+1}} = \frac{1}{(1+\frac{1}{n})^n}\cdot\frac{\frac{a_{n+1}}{(n+1)!}}{\frac{a_n}{n!}} \to \frac{1}{e}\cdot\frac{a}{a} = \frac{1}{e}\quad\text{as } n\to\infty.$$

Thus, by the Cesàro-Lambert lemma, we have $\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty}\sqrt[n]{x_n} = \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \frac{1}{e}$.

Proof of (2): We have

$$\alpha_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} = \frac{\frac{\sqrt[n+1]{a_{n+1}}}{n+1}}{\frac{\sqrt[n]{a_n}}{n}}\cdot\frac{n+1}{n} \to \frac{1/e}{1/e}\cdot 1 = 1.$$

Consequently, we have $\lim_{n\to\infty}\frac{\alpha_n-1}{\ln\alpha_n} = 1$ by the Stolz-Cesàro lemma.

Proof of (3): Noting that

$$\frac{a_{n+1}}{na_n} \;=\; \frac{\frac{a_{n+1}}{(n+1)!}}{\frac{a_n}{n!}} \cdot \frac{n+1}{n} \;\to\; \frac{a}{a} \cdot 1 \;=\; 1,$$

we see that

$$\alpha_n^n \;=\; \left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^n \;=\; \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} \;=\; \frac{a_{n+1}}{na_n} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n}{n+1} \to 1 \cdot e \cdot 1 \;=\; e.$$

This completes the proof.

#1301: *Kenneth B. Davenport, Dallas, PA. Earlier problems in this journal concerned determining closed form solutions to sums of pentagonal numbers (a solution is given in volume 12, number 7, Fall 2007, pages 433–434, problem #1147). Consider more generally the sum of reciprocals of polygonal numbers with an odd number of sides; explicitly, prove*

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{2}{((2n+1)k-(2n-1))k} \\
&= \frac{\pi}{2n-1}\left(\csc\left(\frac{2\pi}{2n+1}\right) - \tan\left(\frac{\pi}{2n+1}\right)\right) + \frac{2\log(4n+2)}{2n-1} \\
&\quad -\frac{4}{2n-1}\sum_{j=1}^{n} \cos\left(\frac{4j\pi}{2n+1}\right) \log\left(\sin\left(\frac{\pi j}{2n+1}\right)\right).
\end{aligned}$$

Solution below by ***Hongwei Chen, Department of Mathematics, Christopher Newport University, Newport News, VA****. Also solved by* ***Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy****.*

Let the infinite series be S. By partial fractions, we have

$$\frac{1}{((2n+1)k-(2n-1))k} \;=\; \frac{1}{2n-1}\left(\frac{2n+1}{(2n+1)k-(2n-1)} - \frac{1}{k}\right)$$

and

$$\begin{aligned}
S \;&=\; \frac{2}{2n-1}\sum_{k=1}^{\infty}\left(\frac{2n+1}{(2n+1)k-(2n-1)} - \frac{1}{k}\right) \\
&=\; \frac{2}{2n-1}\sum_{k=1}^{\infty}\left(\frac{1}{k-\frac{2n-1}{2n+1}} - \frac{1}{k}\right) \\
&=\; \frac{2}{2n-1}\sum_{k=0}^{\infty}\left(\frac{1}{k+\frac{2}{2n+1}} - \frac{1}{k+1}\right).
\end{aligned}$$

In view of the *Digamma function* which is defined by

$$\psi(x) \;:=\; -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{k+x} - \frac{1}{k+1}\right),$$

where γ is the Euler-Mascheroni constant, we have

$$S = -\frac{2}{2n-1}\left(\gamma + \psi\left(\frac{2}{2n+1}\right)\right).$$

Applying the Gauss Digamma formula (for example, see Formula (11) in "Digamma Function": http://mathworld.wolfram.com/DigammaFunction.html)

$$\psi(p/q) = -\gamma - \ln(2q) - \frac{\pi}{2}\cot\left(\frac{p\pi}{q}\right) + 2\sum_{0<j<q/2}\cos\left(\frac{2pj\pi}{q}\right)\ln\left(\sin\left(\frac{j\pi}{q}\right)\right)$$

with $p = 2, q = 2n+1$, we find that

$$S = \frac{2}{2n-1}\left(\ln(4n+2) + \frac{\pi}{2}\cot\left(\frac{2\pi}{2n+1}\right) - 2\sum_{j=1}^{n}\cos\left(\frac{4j\pi}{q}\right)\ln\left(\sin\left(\frac{j\pi}{2n+1}\right)\right)\right).$$

This is equivalent to the claimed result since $\cot\theta = \csc\theta - \tan(\theta/2)$.

Remark. In Chen's *Excursions in Classical Analysis* (MAA, 2010, pp114–117), for $0 < m < k$, instead of applying the Gauss Digamma formula directly, using *Simpson's multi-section formula* and Abel theorem, an explicit formula

$$\begin{aligned} S(k,m) &:= \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{k}{m+kn}\right) \\ &= -\ln k - \frac{\pi}{2}\cot\left(\frac{m\pi}{k}\right) + \frac{1}{2}\sum_{j=1}^{k-1}\cos\left(\frac{2mj\pi}{k}\right)\ln\left(2 - 2\cos\left(\frac{2j\pi}{k}\right)\right) \end{aligned}$$

has been established. In view of the identity that $1 - \cos 2\theta = 2\sin^2\theta$, this recaptured the Gauss Digamma formula. As a consequence, the sum of reciprocals of polygonal numbers with *any* side $r \geq 5$ is given by

$$\sum_{k=1}^{\infty}\frac{2}{((r-2)k-(r-4))k} = -\frac{2}{r-4}S(r-2,2).$$

Here the proposed problem is the the sum of reciprocals of polygonal numbers with $2n+3$ sides.

#1303. *Proposed by E. Ionascu and R. Stephens, Columbus State University, Columbus, GA. Suppose we have K dollars in an account that accumulates compound interest at the rate of $i > 0$ per time period and that a payment of P is made from that account at the end of each time period for n periods such that the balance in the account after the last payment is zero. This is an example of what, in Financial Mathematics, is called an* Ordinary Annunity Certain with

- *Effective Interest Rate i,*
- *Discount Factor $v = \frac{1}{1+i}$, and*
- *Unit Present Value $a = \frac{K}{P} = v + v^2 + \cdots + v^n = \frac{1-v^n}{i}$.*

When $n > 2$, K, and P (and therefore a) are known, it is desirable to determine the interest rate i. Note that if $n = 2$, then the quadratic $a = (1+i)^{-1} + (1+i)^{-2}$ is easily solved for i, but for $n > 2$ it is difficult or impossible to find a closed form exact solution for i in the equation $a = (1+i)^{-1} + \cdots + (1+i)^{-n}$. Various methods for estimating i are well known. For example, $\frac{2(n-a)}{a(n+1)}$ is a good estimate when n is small and $\frac{1-(\frac{a}{n})^2}{a}$ is a good estimate when n is large.

Consider $i_* > 0$ as an estimate for i. Then our estimated Discount Factor is $v_* = \frac{1}{1+i^*}$ and our estimated Unit Present Value is $a_* = v_* + v_*^2 + ... + v_*^n = \frac{1-v_*^n}{i_*}$. Show that

$$i_{**} \;=\; i_* \frac{a_*(\frac{a_*}{a}) - nv_*^{n+1}}{a_* - nv_*^{n+1}}.$$

is positive and a better estimate for i than i_*, in the sense that if $i_* < i$ then $i_{**} > i_*$ and if $i < i_*$ then $i_{**} < i_*$. Note that equation (2.1) appears in JFEP, V. 13, No. 2. The empirical evidence indicates that this significantly improves any reasonable estimate of i, but it is not the case that i_{**} is always between i and i_*. Establishing the conditions under which $|i - i_{**}| < |i - i_*|$ is an open problem.

The first solution was received by ***Mark Evans of Louisville, KY****; the solution below is from the proposers.*

First, we shall show that $i_{**} > 0$. By Bernoulli's inequality $(1+i_*)^n > 1 + ni_*$, which can be written as $1 > v_*^n + ni_*v_*^n$ or $\frac{1-v_*^n}{i_*} > nv_*^n$. So, we have $a_* > nv_*^n > nv_*^{n+1}$. This shows that the denominator in (2.1) is positive: $a_* - nv_*^{n+1} > 0$.

Now, using the AGM inequality, we see that

$$\frac{a_*}{n} \;=\; \frac{v_* + v_*^2 + \cdots + v_*^n}{n} \;>\; (v_*^{\frac{n(n+1)}{2}})^{\frac{1}{n}} \;=\; v_*^{\frac{n+1}{2}},$$

which implies $\frac{a_*^2}{n^2} > v_*^{n+1}$. Then, since we have clearly $a < n$, we get $\frac{a_*}{a} > \frac{a_*}{n}$ or $\frac{a_*}{a}\frac{a_*}{n} > \frac{a_*^2}{n^2} > v_*^{n+1}$. Therefore, we obtain that $a_*(\frac{a_*}{a}) - nv_*^{n+1} > 0$.

Finally, assuming that $i_* < i$, one can easily see that $v_* > v$ which, in turn yields $a_* > a$. Hence, $i_{**} > i_*$. Similarly we can deal with the other case. □

#1305. *Proposed by Steven J. Miller, Williams College, Williamstown, MA. Let $\mathbb{N}_{\text{twin}}$ be the set of all integers whose only prime factors are twin primes (we say p is a twin prime if it is prime and either $p+2$ or $p-2$ is also prime, as except for 2 and 3 all neighboring primes are at least 2 units apart). Thus 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21 and 25 are all in $\mathbb{N}_{\text{twin}}$ while 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 23 and 24 are not. Does*

$$\mathcal{S} \;:=\; \sum_{n \in \mathbb{N}_{\text{twin}}} \frac{1}{n}$$

converge or diverge? If it converges approximate the sum.; if it diverges approximate (as a function of x) $\mathcal{S}(x) := \sum_{n \in \mathbb{N}_{\text{twin}}, n \le x} 1/n$.

Solution below by ***Josiah Banks, Youngstown State University, Youngstown, Ohio.*** *Also solved by* ***Missouri State University Problem Solving Group, Department of Mathematics, Missouri State University***.

First, we note that if $x \geq c \geq 3$, then $x \ln\left(\dfrac{x}{x-1}\right) \leq c \ln\left(\dfrac{c}{c-1}\right)$. This follows from $x \ln\left(\frac{x}{x-1}\right)$ is a decreasing function. Therefore, if $x \geq c \geq 3$, then $x \ln\left(\frac{x}{x-1}\right) \leq c \ln\left(\frac{c}{c-1}\right)$ with equality when $x = c$.

Equivalently, if $x \geq c \geq 3$ then $\ln\left(\dfrac{x}{x-1}\right) \leq \dfrac{c \ln\left(\frac{c}{c-1}\right)}{x}$.

Let $\mathbb{T}_{\mathbb{P}}$ denote the set of all twin primes and $\mathcal{J} = \displaystyle\sum_{p \in \mathbb{T}_{\mathbb{P}}} \ln\left(\frac{p}{p-1}\right)$. Then $\mathcal{J} < \infty$, in fact $\mathcal{J} \in (2.018, 2.036)$.

To see this, recall the sum of the reciprocals of the twin primes converges to Brun's constant (about 1.90216), and notice

$$\begin{aligned}
\mathcal{J} &= \sum_{p \in \mathbb{T}_{\mathbb{P}}} \ln\left(\frac{p}{p-1}\right) \\
&\leq \sum_{p \in \mathbb{T}_{\mathbb{P}}} \frac{3 \ln\left(\frac{3}{2}\right)}{p} \quad \leftarrow \text{(by our first fact)} \\
&= \ln\left(\frac{3}{2}\right) \mathcal{B} \quad \leftarrow \text{(where } \mathcal{B} \text{ is Brun's constant)} \\
&< \infty.
\end{aligned}$$

Therefore, $\mathcal{J} < \infty$.

To approximate $\mathcal{J}$ let $M = \{3, 5, 7, 11, 13, 17, 19\}$ and we notice

$$\begin{aligned}
\mathcal{J} &= \sum_{p \in \mathbb{T}_{\mathbb{P}}} \ln\left(\frac{p}{p-1}\right) = \sum_{p \in \mathbb{T}_{\mathbb{P}}} -\ln(1 - \frac{1}{p}) \\
&= \sum_{p \in \mathbb{T}_{\mathbb{P}}} \sum_{k=1}^{\infty} \frac{1}{kp^k} = \mathcal{B} + \sum_{p \in \mathbb{T}_{\mathbb{P}}} \sum_{k=2}^{\infty} \frac{1}{kp^k} \\
&> \mathcal{B} + \sum_{p \in \mathbb{T}_{\mathbb{P}}} \sum_{k=2}^{4} \frac{1}{kp^k} > \mathcal{B} + \sum_{p \in M} \sum_{k=2}^{4} \frac{1}{kp^k} > 2.018.
\end{aligned}$$

So $\mathcal{J} > 2.018$.

With M as above, then for the upper bound we observe that

$$\begin{aligned}
\mathcal{J} &= \sum_{p\in\mathbb{T}_{\mathbb{P}}} \ln\left(\frac{p}{p-1}\right) \\
&= \ln\left(\frac{3}{2}\right)+\ln\left(\frac{5}{4}\right)+\ln\left(\frac{7}{6}\right)+\ln\left(\frac{11}{10}\right)+\ln\left(\frac{13}{12}\right)+\ln\left(\frac{17}{16}\right)+\ln\left(\frac{19}{18}\right)+\sum_{p\in\mathbb{T}_{\mathbb{P}}\setminus M}\ln\left(\frac{p}{p-1}\right) \\
&= \ln\left(\frac{323323}{110592}\right)+\sum_{p\in\mathbb{T}_{\mathbb{P}}\setminus M}\ln\left(\frac{p}{p-1}\right) \\
&\leq \ln\left(\frac{323323}{110592}\right)+\sum_{p\in\mathbb{T}_{\mathbb{P}}\setminus M}\frac{29\ln\left(\frac{29}{28}\right)}{p} \\
&= \ln\left(\frac{323323}{110592}\right)+29\ln\left(\frac{29}{28}\right)\left(\mathcal{B}-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}-\frac{1}{17}-\frac{1}{19}\right) < 2.036.
\end{aligned}$$

Therefore we have that $\mathcal{J}$ is within the interval $(2.018, 2.036)$.

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