# PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2016 

STEVEN J. MILLER (EDITOR)

## 1. Problems: Spring 2016

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk $\left({ }^{*}\right)$ preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.
\#1312: Proposed by Steven J. Miller, Department of Mathematics and Statistics, Williams College, Williamstown, MA and Stan Wagon, Department of Mathematics, Statistics, and Computer Science, Macalester College, St. Paul, Minnesota.

Larry Bird and Magic Johnson are playing a game of basketball; they alternating shooting with Bird going first, and the first to make a basket wins. Assume Bird always makes a basket with probability $p_{B}$ and Magic with probability $p_{M}$, where $p_{B}$ and $p_{M}$ are independent uniform random variables. (This means the probability each of them is in $[a, b] \subset[0,1]$ is $b-a$, and knowledge of the value of $p_{B}$ gives no information on the value of $p_{M}$.)
(1) What is the probability Bird wins the game?
(2) What is the probability that, when they play, Bird has as good or greater chance of winning than Magic?
Note: for another related problem, see Math Horizons 23:2 (Nov 2015), 30.
\#1313: Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.

Suppose that $a b+b c+c a=8 a b c$ and $a, b, c \geq 1 / 5$. Prove that $a^{2}+b^{2}+c^{2}+15 a b c>$ $15 a^{2} b c+15 a b^{2} c+15 a b c^{2}$. Furthermore prove that for any positive constant $\epsilon$ the inequality $a^{2}+b^{2}+c^{2}+15 a b c>15 a^{2} b c+15 a b^{2} c+15 a b c^{2}+\epsilon a^{2} b^{2} c^{2}$ does not hold for all $a, b, c \geq 1 / 5$.

[^0]\#1314: Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.
The following problem is an expanded version of a problem from the 2013 Green Chicken Math Competition between Middlebury and Williams College. Show that there is a positive Fibonacci number that is divisible by 1000 , and find the smallest such number; more generally, find all positive integers $N$ such that there exists a positive Fibonacci number $F_{n}$ which is divisibly by $N$, and for such $N$ give a bound for the smallest index $n$ that works in terms of $N$.
\#1315: Proposed by Steven J. Miller, Williams College, and Daniel F. Stone, Bowdoin College.

Let

$$
b_{n}=a\left(b+\frac{1}{n} \sum_{i=1}^{n-1} b_{i}\right) \text { for } n>1, \text { with } b_{1}=a b
$$

where $b, a>0$.
(1) Show that if $a=1$, then $b_{n}=H_{n}=1+1 / 2+1 / 3+\cdots+1 / n$ is the $n^{\text {th }}$ harmonic number, and thus diverges. Determine the rate of growth of $b_{n}$ as a function of $n$.
(2) Show that if $a<1$, then $b_{n}$ is bounded and determine the best bound possible.

This problem is motivated by research of the second proposer on political polarization, who proposes a simple mathematical model of affective polarization in which two opposing partisans Bayesian update their beliefs about the opposition's pro-sociality, i.e., degree to which she is willing to sacrifice a personal interest for the greater good. The paper shows that two seemingly unrelated cognitive biases can cause highly distorted, polarized, beliefs about this characteristic for the opposition. One of these biases is called the false consensus bias, which means under-estimation of the extent to which different people have different tastes. Think about how when you're in the mood for ice cream, it's hard to imagine someone wanting a salad. But some people genuinely do prefer salad then, and indeed, you are probably under-estimating the chance of this. Stone shows this misperception can cause actions that are consistent with tastes about what is best for the greater good to appear to be self-serving and thus make one less liked by the opposition, causing affective polarization. He shows a lower-bound for the expectation of such polarization after $t$ periods of interaction is our $b_{n}$, in which $b$ is the initial bias (under-estimation of difference in mean tastes for the two parties) and $a$ represents the degree to which new information affects beliefs (this is a function of other parameters of the model).
\#1316: Proposed by Mehtaab Sawhney, Commack High School.
Consider an $n \times n$ chessboard for $n \geq 2$. Define a left-rook to be a rook that can only attack the squares in the same row to its left. Similarly define right-rooks, up-rooks, and down-rooks. Find the maximum total of right-rooks, left-rooks, up-rooks, and down-rooks, as a function of $n$, such that no rook is attacking another.
\#1317: Robert C. Gebhardt, Chester, NJ.
(a) A continuous differentiable function $g(x)>0, a<x<b$, is revolved about the $x$-axis to create a surface of revolution with area $S$, and a volume of revolution $V$. Find all functions $g(x)$, other than $g(x)=0$ and $g(x)=2$, such that the surface area $S$ (in square units) is the
same number as the volume $V$ (in cubic units) for all finite choices of $a$ and $b$ (the areas of the discs at each end of the volume are not included).
(b) Let $f(x)$ and $g(x)$ be continuous differentiable functions, $a<x<b$, where $0<f(x)<$ $g(x)$. Each is revolved about the $x$-axis to create a volume between them. Find all such functions $f(x)$ and $g(x)$ such that the surfaces' total area $S$ (in square units) and the volume $V$ (in cubic units) are the same number. The washers at each end of the volume are not included.

## 2. Solutions

\#1306: Proposed by David Vella, Mathematics and Computer Science Deptartment, Skidmore College, Saratoga Springs, NY 12866.
Find all integer solutions $(p, q)$ to the equation

$$
q^{p+q}+p^{p}(p+q)^{p}=\left(p^{2}+q\right)^{q},
$$

where $p$ and $q$ are prime numbers.
Solution below by Hongwei Chen, Department of Mathematics, Christopher Newport University, Newport News, VA 23606. Also solved by Panagiotis T. Krasopoulos, Social Insurance Institute, Athens, Greece, Khanh Le, Ohio Wesleyan University, Benjamin G. Klein, Department of Mathematics, Davidson College, Davidson, NC.
Solution. We show that the only solution is $(p, q)=(2,3)$. First direct computation shows that $p=q=2$ is not a solution. Next, notice that if $p, q>2$ then $2 \mid(p+q)$ and $2 \mid\left(p^{2}+q\right)$, and so $2 \mid q$. This contradicts $q>2$ is prime. Similarly, if $p=q>2$, in view of that $2 \mid(p+q)$ and $2 \mid\left(p^{2}+p\right)=p(p+1)$, it follows that $2 \mid q$ again, giving us the same contradiction as before. Thus, we only need to consider two cases:

$$
\text { (i) } q=2 \text { and } p>2, \quad \text { (ii) } p=2 \text { and } q>2 \text {. }
$$

(i). When $q=2$ and $p>2$, (1) becomes

$$
4 \cdot 2^{p}+p^{p}(p+2)^{p}=\left(p^{2}+2\right)^{2} .
$$

But, this is impossible because $p^{p}(p+2)^{p}=\left(p^{2}+2 p\right)^{p}>\left(p^{2}+2\right)^{2}$, and thus there is no solution to the equation in this case.
(ii). When $p=2$ and $q>2$, (1) becomes

$$
q^{2+q}+4(2+q)^{2}=(4+q)^{q}
$$

or $16 \equiv 4^{q}(\bmod q)$. By Fermat's little Theorem, if $a$ is relatively prime to a prime $q$ then $a^{q-1} \equiv 1(\bmod q)$. Hence $16 \equiv 4^{q}(\bmod q)$ is equivalent to $4 \equiv 1(\bmod q)$, which only holds for $q=3$ only. Direct computation verifies that $p=2, q=3$ works.

In summary, we have proved that $(p, q)=(2,3)$ is the only solution as claimed.
\#1307: Proposed by Panagiotis T. Krasopoulos, Social Insurance Institute, Athens, Greece.

Let $p(x)$ be a polynomial with complex coefficients of degree $n \geq 2$ with distinct roots $\alpha_{1}, \ldots, \alpha_{n}$ and let $p^{\prime}(z)$ be its derivative. Prove elementarily (i.e., do not use contour integration and complex analysis) that

$$
\sum_{k=1}^{n} \frac{1}{p^{\prime}\left(\alpha_{k}\right)}=0
$$

First solution below by Benjamin G. Klein, Davidson College. Also solved by Henry Ricardo, New York Math Circle (whose solution is also given as it arrived essentially simultaneously and is beautifully written), J. Sorel, Romania, Sayok Chakravarty, Troy High School, Fullerton, CA, Hongwei Chen, Department of Mathematics, Christopher Newport University, Khanh Le, Ohio Wesleyan University, Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.

First Solution: Ben Klein:
Since $p$ is a polynomial of degree $n$ with $n$ distinct roots, we know that $p(z)=a\left(\prod_{k=1}^{n}\left(z-\alpha_{k}\right)\right)$ where $a$ is a non-zero complex constant. Then the rational function

$$
\frac{1}{p(z)}=\frac{1 / a}{\left(z-\alpha_{1}\right) \cdots\left(z-\alpha_{n}\right)}
$$

has a partial fraction decomposition of the form

$$
\frac{1}{p(z)}=\sum_{k=1}^{n} \frac{A_{k}}{z-\alpha_{k}}
$$

where the $A_{k}$ are complex constants. Since, for $z \neq \alpha_{1}$,

$$
\frac{z-\alpha_{1}}{p(z)}=\frac{z-\alpha_{1}}{p(z)-0}=\frac{z-\alpha_{1}}{p(z)-p\left(\alpha_{1}\right)}=\sum_{k=1}^{n} \frac{A_{k}\left(z-\alpha_{1}\right)}{z-\alpha_{1}}=A_{1}+\sum_{k=2}^{n} \frac{A_{k}\left(z-\alpha_{1}\right)}{z-\alpha_{1}},
$$

we can take the limit as $z \rightarrow \alpha_{1}$ and discover that $A_{1}=1 / p^{\prime}\left(\alpha_{1}\right)$. Similarly, $A_{k}=1 / p^{\prime}\left(\alpha_{k}\right)$ for $k=2, \ldots, n$. (We note that since the roots of $p(z)$ are distinct, no $p^{\prime}\left(\alpha_{k}\right)$ is zero.)

Now

$$
\frac{z}{p(z)}=\sum_{k=1}^{n} \frac{\frac{z}{p^{\prime}\left(\alpha_{k}\right)}}{z-\alpha_{k}}=\sum_{k=1}^{n} \frac{z}{z-\alpha_{k}} \frac{1}{p^{\prime}\left(\alpha_{k}\right)} .
$$

If we let $z \rightarrow \infty$ and recall that the degree of $p(z)$ is at least two, we have $z / p(z) \rightarrow 0$ and

$$
\sum_{k=1}^{n} \frac{z}{z-\alpha_{k}} \frac{1}{p^{\prime}\left(\alpha_{k}\right)} \rightarrow \sum_{k=1}^{n} \frac{1}{p^{\prime}\left(\alpha_{k}\right)}
$$

establishing the result.
Comments from the solver: This problem and generalizations of it can be found in PolyaSzego's Problems and Theorems in Analysis, but, as I recall, the solution there involves methods from complex analysis. I discovered this solution after stumbling on the result myself while teaching a course in complex analysis at Davidson. The elementary argument above is, I believe, the same argument that was given by Euler, as reported by William Dunham in a talk at a national MAA-AMS meeting. I did not attend this meeting but
learned about the argument from a colleague here at Davidson who did attend. My colleague knew I would be interested in an elementary argument since he and I had talked about the result and he knew that I had a 'non-elementary' argument.

Second Solution: Henry Ricardo:
Since $p(z)$ has distinct roots, we have the partial fraction decomposition

$$
\frac{1}{p(z)}=\sum_{k=1}^{n} \frac{c_{k}}{z-\alpha_{k}}
$$

Then for any $j, 1 \leq j \leq n$, we see that $p(z)=\left(z-\alpha_{j}\right) q_{j}(z)$, where $q_{j}$ is of degree $n-1$, and

$$
\begin{equation*}
\frac{1}{q_{j}(z)}=\frac{z-\alpha_{j}}{p(z)}=\sum_{k=1}^{n} \frac{c_{k}\left(z-\alpha_{j}\right)}{z-\alpha_{k}} \tag{*}
\end{equation*}
$$

Since $p^{\prime}\left(\alpha_{j}\right)=q_{j}\left(\alpha_{j}\right)$, it follows that $c_{j}=1 / p^{\prime}\left(\alpha_{j}\right)$. Therefore, multiplying each member of $(*)$ by $q_{j}(z)$, we find that

$$
1=\sum_{k=1}^{n} c_{k} \frac{p(z)}{z-\alpha_{k}}=\sum_{k=1}^{n} \frac{1}{p^{\prime}\left(\alpha_{k}\right)} q_{k}(z),
$$

where each summand is a polynomial of degree $n-1$ whose leading coefficient is the leading coefficient of $p(z)$. This implies that the sum is of degree $n-1$; and, since the sum is constant, the coefficient of $z^{n-1}$ must be 0 - that is,

$$
\sum_{k=1}^{n} \frac{1}{p^{\prime}\left(\alpha_{k}\right)}=0
$$

\#1308: Proposed by Taimur Khalid, Coral Academy of Science LV.
Consider a triangle $A B C$. Let the external angle bisectors of angles $A$ and $B$ intersect at a point $D, B$ and $C$ at $E$, and $A$ and $C$ at $F$. See Figure 1 .
(1) Prove that the circumcircles of triangles $A D B, B E C$, and $C F A$ intersect at a common point.
(2) Prove that this point is the incenter of $\triangle A B C$.

Solution below by by Ioana Mihăilă, Cal Poly Pomona. Also solved by Shreya Dalal and Tommy Goebeler, The Episcopal Academy, Newtown, Square, PA, and the Skidmore College Problem Group, Saratoga Springs, NY.

Let I be the incenter of $\triangle A B C$. Since from the hypothesis $A D$ is the external angle bisector of angle $A, \angle D A B=\left(180^{\circ}-\angle A\right) / 2$. By construction $A I$ is the interior bisector of angle $A$ (the incenter of a triangle is the intersection of the angle bisectors), so $\angle I A B=\angle A / 2$. Therefore $\angle D A I=\angle D A B+\angle I A B$ is a right angle. Similarly $\angle D B I$ is a right angle.

A polygon is said to be cyclic if all its vertices lie on a circle. Recall that a quadrilateral is cyclic if and only if its opposite angles add up to $180^{\circ}$. Thus the quadrilateral $D A I B$ is cyclic since $\angle D A I+\angle D B I=9^{\circ}+90^{\circ}=180^{\circ}$. This means that the points $D, A, B$, and $I$


Figure 1. Triangle $A B C$ and its external angle bisectors.
lie on a circle, and since the circumcircle of $A D B$ is unique, it must go through the incenter $I$.

Similarly, the circumcircles of triangles $B E C$ and $C F A$ also pass through $I$, thus $I=T$ is the common intersection point of the three circumcircles.
\#1309: Proposed by Kenneth B. Davenport, Dallas, PA.
The Chebyshev polynomials are defined recursively by $T_{N+1}(x)=2 x T_{N}(x)-T_{N-1}(x)$ for $N \geq 1$, with $T_{0}(x)=1, T_{1}(x)=x$ (and thus $T_{2}(x)=2 x^{2}-1$ and $T_{3}(x)=4 x^{3}-3 x$ ). They have many applications in mathematics, especially in approximation theory and polynomial interpolation. As they are of the form $T_{N}(x)=\cos (N \arccos (x))$, it is interesting to look at cosines (and hence also sines) of arccosines of angles. Prove

$$
(-1)^{N} \cos (N \theta)=\cos (2 N \psi), \quad(-1)^{N+1} \sin (N \theta)=\sin (2 N \psi),
$$

where

$$
\theta=\arccos \left(\frac{x}{\sqrt{x^{2}+4}}\right), \quad \psi=\arctan \left(\frac{x+\sqrt{x^{2}+4}}{2}\right) .
$$

Solution below by Hongwei Chen, Department of Mathematics, Christopher Newport University, Newport News, VA 23606. Also solved by Panagiotis T. Krasopoulos, Social Insurance Institute, Athens, Greece, Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

In the arguments below we use several properties of these polynomials, including the composition identity (or nesting property) that $T_{m n}(x)=T_{m}\left(T_{n}(x)\right)$ and that $T_{n}(-x)=$ $(-1)^{n} T_{n}(x)$.

Since

$$
\begin{aligned}
\cos (2 N \psi) & =T_{2 N}(\cos \psi)=T_{N}\left(T_{2}(\cos \psi)\right) \\
& =T_{N}\left(2 \cos ^{2} \psi-1\right)=(-1)^{N} T_{N}\left(1-2 \cos ^{2} \psi\right),
\end{aligned}
$$

in view of the fact that $T_{N}(\cos \theta)=\cos (N \theta)$, it suffices to show that

$$
\begin{equation*}
1-2 \cos ^{2} \psi=\cos \theta \tag{1}
\end{equation*}
$$

Indeed, we have

$$
\begin{aligned}
1-2 \cos ^{2} \psi & =1-\frac{2}{\sec ^{2} \psi}=\frac{\tan ^{2} \psi-1}{\tan ^{2} \psi+1} \\
& =\frac{\left(\frac{x+\sqrt{x^{2}+4}}{2}\right)^{2}-1}{\left(\frac{x+\sqrt{x^{2}+4}}{2}\right)^{2}+1}=\frac{x}{\sqrt{x^{2}+4}}
\end{aligned}
$$

This proves (1) as desired from the definition of $\theta$, and so

$$
\begin{equation*}
(-1)^{N} \cos (N \theta)=\cos (2 N \psi) \tag{2}
\end{equation*}
$$

Next, differentiating (2) with respective to $x$ yields

$$
\begin{equation*}
(-1)^{N} N \sin (N \theta) \frac{d \theta}{d x}=2 N \sin (2 N \psi) \frac{d \psi}{d x} . \tag{3}
\end{equation*}
$$

Since

$$
\frac{d \theta}{d x}=-\frac{2}{x^{2}+4}, \quad \frac{d \psi}{d x}=\frac{1}{x^{2}+4},
$$

substituting them into (3), we find that

$$
(-1)^{N+1} \sin (N \theta)=\sin (2 N \psi)
$$

\#1311: Proposed by Abdilkadir Altintaş, Emirdă̆, Afyon, Turkey.
Compute the product

$$
\left(\frac{1}{\sqrt{3}}+\tan 59^{\circ}\right)\left(\frac{1}{\sqrt{3}}+\tan 58^{\circ}\right) \cdots\left(\frac{1}{\sqrt{3}}+\tan 2^{\circ}\right)\left(\frac{1}{\sqrt{3}}+\tan 1^{\circ}\right) .
$$

Solution below by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Also solved by Hongwei Chen, Department of Mathematics, Christopher Newport University, Newport News, VA 23606, Robert Gebhardt, Chester, NJ, Alan Levine, Department of Mathematics, Franklin and Marshall College, Lancaster, PA 17604, Panagiotis T. Krasopoulos, Social Insurance Institute, Athens, Greece, Kenneth B. Davenport, Dallas, PA, Khanh Le, Ohio Wesleyan University, Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, Shreya Dalal, The Episcopal Academy, Newtown Square, PA, the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO, and the Skidmore College Problem Group, Saratoga Springs, NY.

The given product is equal to $\frac{2}{\sqrt{3}}\left(\frac{4}{3}\right)^{29} \approx 4849.5$, which follows considering the product of the terms of the form

$$
p_{k}=\left(\frac{1}{\sqrt{3}}+\tan (30+k)^{\circ}\right)\left(\frac{1}{\sqrt{3}}+\tan (30-k)^{\circ}\right) .
$$

For simplicity, let $a=\frac{1}{\sqrt{3}}=\tan 30^{\circ}$, and $b=\tan k^{\circ}$, then

$$
\begin{aligned}
p_{k} & =\left(a+\frac{a+b}{1-a b}\right)\left(a+\frac{a-b}{1+a b}\right) \\
& =\frac{b^{2}+a^{4} b^{2}-2 a^{2}\left(2+b^{2}\right)}{-1+a^{2} b^{2}} .
\end{aligned}
$$

Now, since $a^{2}=1 / 3$ we get

$$
p_{k}=\frac{\frac{10 b^{2}}{9}-\frac{2}{3}\left(2+b^{2}\right)}{-1+\frac{b^{2}}{3}}=\frac{4}{3},
$$

from which the result follows.

## Other notes:

\#1300: A correct solution was also received by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA; see the Fall 2015 issue for a solution.
\#1305: (The last few lines in the solution by Josiah Banks, Youngstown State University, Youngstown, Ohio, were accidentally left out, and are included below.) Proposed by Steven J. Miller, Williams College, Willamstown, MA. Let $\mathbb{N}_{\text {twin }}$ be the set of all integers whose only prime factors are twin primes (we say $p$ is a twin prime if it is a prime and either $p+2$ or $p-2$ is also a prime, as except for 2 and 3 all neighboring primes are at least 2 units apart). Thus $1,3,5,7,9,11,13,15,17,19,21$, and 25 are all in $\mathbb{N}_{\text {twin }}$ while $2,4,6,8,10,12,14,16,18,20,22,23$, and 24 are not. Does

$$
\mathcal{S}:=\sum_{n \in \mathbb{N}_{\mathrm{twin}}} \frac{1}{n}
$$

converge or diverge? If it converges approximate the sum; if it diverges approximate it (as a function of $x) \mathcal{S}(x):=\sum_{n \in \mathbb{N}_{\text {twin }}, n \leq x} \frac{1}{n}$.

The solution from Banks started with

$$
\begin{aligned}
\mathcal{J} & =\sum_{p \in \mathbb{T}_{\mathbb{P}}} \ln \left(\frac{p}{p-1}\right) \\
& \leq \sum_{p \in \mathbb{T}_{\mathbb{P}}} \frac{3 \ln \left(\frac{3}{2}\right)}{p} \\
& =3 \ln \left(\frac{3}{2}\right) \mathcal{B} \leftarrow \text { (where } \mathcal{B} \text { is Brun's constant) } \\
& <\infty,
\end{aligned}
$$

and ended by showing that $\mathcal{J}$ is within the interval $(2.018,2.036)$.
The following is the omitted text. The proof is concluded by noting $\mathcal{S}=e^{\mathcal{J}}$, where $\mathcal{J}=\sum_{p \in \mathbb{T}_{\mathbb{P}}} \ln \left(\frac{p}{p-1}\right)$ and thus an approximation of $\mathcal{S}$ is $7.52<\mathcal{S}<7.66$. To see this, notice

$$
\mathcal{S}=\sum_{n \in \mathbb{N}_{\text {twin }}} \frac{1}{n}=\prod_{p \in \mathbb{T}_{\mathbb{P}}}\left(\frac{p}{p-1}\right)=e^{\sum_{p \in \mathbb{T}_{\mathbb{P}}} \ln \left(\frac{p}{p-1}\right)}=e^{\mathcal{J}} .
$$

As $\mathcal{J} \in(2.018,2.036)$, it follows that $\mathcal{S} \in\left(e^{2.018}, e^{2.036}\right)$ which implies $\mathcal{S} \in(7.52,7.66)$.

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[^0]:    Date: March 16, 2016.

