PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2016

STEVEN J. MILLER (EDITOR)

1. Problems: Fall 2016

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.

#1318: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*

The following is from the 2009 Green Chicken Math Competition between Middlebury and Williams Colleges. Evaluate

$$\sin(2\pi/3^n)\sin(\pi/3^n).$$

#1319: *Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.*

A classic linear algebra problem is to calculate the determinant of the symmetric Pascal Matrix, P, which is the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is $P_{i,j} = \binom{i+j}{j}$. Notice that the Pascal Matrix satisfies $P_{i,j} = P_{i-1,j} + P_{i,j-1}$ for i and j greater than 1, which is equivalent to Pascal's Identity. Building on this identity, we can consider the more general family of matrices $A(n)$ such that $A(n)$ is an $n \times n$ matrix whose entries satisfy

$$A(n)_{i,j} \ = \ \begin{cases} 1 & i = 1 \text{ and/or } j = 1 \\ A(n)_{i-1,j} + A(n)_{i,j-1} + kA(n)_{i-1,j-1} & \text{otherwise;} \end{cases}$$

notice that the symmetric Pascal Matrix is simply the case $k = 0$. Determine $\det(A(n))$ as a function of k and n.

Date: November 16, 2016.

Hint: It is known that the Pascal Matrix has an LU factorization as a consequence of Vandermonde's Identity. One approach (although not the easiest) for this problem is to show that $A(n)$ has an LU factorization and then use this factorization to compute the determinant.

#1320: *Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.*

There are numerous inequalities relating real numbers which may be studied. The one below is a little non-standard in that it has a non-symmetric choice of inputs where equality holds, indicating that some standard techniques will not suffice for a full analysis.

Prove that

$$x^5 + y^5 + z^5 + 2(\sqrt{2}-1)xyz(xy + yz + xz) \ \geq \ \frac{2\sqrt{2}-1}{2}(x^4y + x^4z + y^4x + y^4z + z^4x + z^4y)$$

for all nonnegative real numbers x, y, and z, and show that there exists a triple (x, y, z) of non-zero real numbers such that (1) the inequality is an equality, and (2) the three numbers are not identical.

#1321: *Proposed by Steven J. Miller, Williams College, Williamstown, MA 01267.*

In 1742 Christian Goldbach wrote a letter to Leonard Euler with conjectures on writing integers as the sum of primes. The modern formulation is to split the question, with the binary Goldbach problem being the statement that every sufficiently large even number is the sum of two primes and the ternary Goldbach problem every sufficiently large odd number is the sum of three primes. It is believed that 'sufficiently large' means at least 4 for the binary problem, though this is far from proved. The situation is very different in the odd case. It has long been known to be true, and recent work of Harald Helfgott finished the argument and showed that 'sufficiently large' here is at least 7; as an immediate consequence we obtain every even number is the sum of at most 4 primes. The chain of ideas leading to these results use the Circle Method, one of the most beautiful but also one of the most technical arguments in number theory. It is often worthwhile seeing what can be proved with weaker inputs. Specifically, prove `elementarily` that if $x \geq 2$ is a positive integer, then we may write x as a sum of at most $\log_2(x)$ primes.

#1322: *Proposed by Gabriel Prajitura, SUNY Brockport, Brockport, NY 14420*

Construct a sequence of integers $(x_n)_n$ such that the set

$$\left\{x_1,\ \frac{x_1+x_2}{2},\ \frac{x_1+x_2+x_3}{3},\ \dots\right\}$$

is dense in $\mathbb{R}$.

#1323: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*

The following is from the 1997 Green Chicken Math Competitifon between Middlebury and Williams Colleges. Does any row of Pascals triangle have three consecutive entries that in the ratio 1:2:3?

#1324: *Proposed by Mehtaab Sawhney, University of Pennsylvania, Philadelphia, PA.*

The Cauchy-Schwarz Inequality, one of the most important and most used in mathematics, states that

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \ \geq \ \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

One natural attempt to generalize to a multidimensional sum is

$$\left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} a_\ell a_k\right)\left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} b_\ell b_k\right) \ \geq \ \left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} a_\ell b_k\right)^2,$$

but a little inspection shows that this is actually just an equality. This suggests trying a weighted version of the above:

$$\left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} c_{k+\ell} a_\ell a_k\right)\left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} c_{k+\ell} b_\ell b_k\right) \ \geq \ \left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} c_{k+\ell} a_\ell b_k\right)^2.$$

Part (1) is equivalent to $c_{k+\ell} = (k+\ell)!$ while part (2) is $c_{k+\ell} = \frac{1}{k+\ell}$.

(1) Prove that

$$\left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} (k+\ell)! a_\ell a_k\right)\left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} (k+\ell)! b_\ell b_k\right) \ \geq \ \left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} (k+\ell)! a_\ell b_k\right)^2.$$

(2) Prove that

$$\left(\sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{1}{k+\ell} a_\ell a_k\right)\left(\sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{1}{k+\ell} b_\ell b_k\right) \geq \left(\sum_{k=1}^{n}\sum_{\ell=1}^{n} \frac{1}{k+\ell} a_\ell b_k\right)^2.$$

(3) (Open) Under what conditions for c_j does the inequality

$$\left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} c_{k+\ell} a_\ell a_k\right)\left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} c_{k+\ell} b_\ell b_k\right) \ \geq \ \left(\sum_{k=0}^{n}\sum_{\ell=0}^{n} c_{k+\ell} a_\ell b_k\right)^2$$

hold? Note that $c_j > 0$ is neither a necessary nor sufficient condition! (In particular for $n = 1$ it can be shown that $c_0 = 2$, $c_1 = -1$, and $c_2 = 2$ works while $c_0 = 2$, $c_1 = 3$, and $c_2 = 1$ doesn't.)

#1325: *Proposed by Matthew McMullen, Otterbein U., Westerville, OH.*
An equable triangle is a triangle whose area is numerically equal to its perimeter. Find infinitely many (or, better yet, *all*) right, equable triangles with rational side lengths.

#1326: *Proposed by Steven J. Miller, Williams College, Williamstown, MA.*
For each positive integer n consider an $n \times n$ chessboard, and consider all possible ways of placing n queens on the board; remember a queen can attack anything in her row, column, or along diagonals. If $\mathcal{C}$ is one such placement, let $p(n;\mathcal{C})$ be the number of pawns which can safely be placed on the board without being attacked by one of the queens, and set $p(n) = \max_{\mathcal{C}} p(n;\mathcal{C})$; thus $p(n) = 0$ for $n \leq 3$, $p(4) = 1$ and $p(5) = 3$. Prove there exist positive constants c_1, c_2, δ such that for all n sufficiently large, $n - c_2 n^{1-\delta} \leq p(n) \leq n - c_1 n^{1-\delta}$.

In particular, this means as n grows there is a choice of queen placement so that almost all squares are safe for pawns!

2. Solutions

Note: At the 2016 Math Camp at the College of New Jersey, run by Dan Flegler and Steve Conrad, the problems from the previous issue were given to the participants as a challenge. Honglin Zhu, currently at Eaglebrook School in MA and previously at the Tsinghua University High School in China, is commended for solving the more of these than others (he solved #1313 and #1316).

#1313: *Proposed by Mehtaab Sawhney, Commack High School.*
Suppose that $ab + bc + ca = 8abc$ and $a, b, c \geq 1/5$. Prove that

$$a^2 + b^2 + c^2 + 15abc > 15a^2bc + 15ab^2c + 15abc^2.$$

Furthermore prove that for any positive constant ϵ the inequality

$$a^2 + b^2 + c^2 + 15abc > 15a^2bc + 15ab^2c + 15abc^2 + \epsilon\, a^2b^2c^2$$

does not hold for all $a, b, c \geq 1/5$.

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does not hold for all $a, b, c \geq 1/5$.

Solution below by ***Brian D. Beasley, Department of Mathematics,Presbyterian College, Clinton, SC 29325***.

(i) Without loss of generality, assume $ab + bc + ca = 8abc$ with $a \geq b \geq c \geq 1/5$. Let $u = 1/a$, $v = 1/b$, and $w = 1/c$. Then $0 < u \leq v \leq w \leq 5$ with $u + v + w = 8$, and the given inequality is equivalent to

$$v^2w^2 + u^2w^2 + u^2v^2 + 15uvw \ > \ 15(vw + uw + uv),$$

or $w^2(u + v)^2 - 15w(u + v) > uv(15 - 15w + 2w^2 - uv)$. Let $m = u + v$ and $n = uv$, so that $w = 8 - m$. Substituting and simplifying, we must then prove that

$$m(m - 3)(m - 5)(m - 8) \ > \ n(2m^2 - 17m + 23 - n),$$

where $3 \leq m \leq 16/3$. Let $f(m) = m(m - 3)(m - 5)(m - 8)$ and $g(m) = 2m^2 - 17m + 23$. We note that $g(m) < 0$ for $3 \leq m \leq 16/3$. Thus for $3 \leq m \leq 5$, we have $f(m) \geq 0$ and $n(g(m) - n) < 0$. Next, for $5 < m \leq 16/3$, since f is decreasing on $(5, 16/3]$, we have $f(m) \geq f(16/3) = -896/81$; since $0 < u \leq v \leq w$ and $u + v > 5$, we have $v < 4$ and hence $u > 1$, so $uv = u(m-u) > m-1 > 4$ and thus $n(g(m)-n) < -16$. Hence for $3 \leq m \leq 16/3$, we conclude $f(m) > n(g(m) - n)$ as needed.

(ii) Given any $\epsilon > 0$, we seek $0 < u \leq v \leq w \leq 5$ with

$$v^2w^2 + u^2w^2 + u^2v^2 + 15uvw \ \leq \ 15(vw + uw + uv) + \epsilon,$$

or equivalently $f(m) \leq n(g(m)-n)+\epsilon$. Let $h(x) = x(x+2)(x-3)(x-5)$. Since $h(x) > 0$ for $0 < x \leq 3/2$ and $\lim_{x\to 0^+} h(x) = 0$, we may choose δ with $0 < \delta \leq 3/2$ such that $h(\delta) < \epsilon$.

Then we take $u = \delta$, $v = 3 - \delta$, and $w = 5$. For these choices, we note that $f(m) = 0$ and $n(g(m) - n) + \epsilon = -h(\delta) + \epsilon > 0$, so the result follows.

Addendum. In arguing that $uv > 4$ in the last case of part (i), we are using the fact that for a fixed value of m in $(5, 16/3]$, since $1 < u \leq m/2$, the function $q(u) = u(m-u)$ is increasing on $(1, m/2]$ and hence must be greater than $q(1)$.

#1316: *Proposed by Mehtaab Sawhney, Commack High School.*

Consider an $n \times n$ chessboard for $n \geq 2$. Define a left-rook to be a rook that can only attack the squares in the same row to its left. Similarly define right-rooks, up-rooks, and down-rooks. Find the maximum total of right-rooks, left-rooks, up-rooks, and down-rooks, as a function of n, such that no rook is attacking another.

Solution below by ***Maddi Guillaume, Ben Byrd, and Mikayla Schultz,Taylor University, Upland, IN****, and* ***Mark Evans, Louisville, KY****. Also solved by* ***Ioana Mihaila, Cal Poly Pomona****, the* ***Ashland University Undergraduate Problem Solving Group, Ashland University, Ashland, OH****, and* ***Dax Jantz, North Central College, Naperville, IL****.*

It is possible to have $4n - 4$ rooks by placing appropriate ones along the perimeter (put up rooks in the top row, down rooks in the bottom row, and then left rooks in the remaining spots on the left column and right rooks in the remaining right column). We must show that it is impossible to have $4n - 3$ or more rooks.

Assume that there is a configuration with $4n-3$ rooks and this is the maximum number of rooks that may be placed (the argument proceeds identically for the other three possibilities, $4k - 2, 4k - 1$ and $4k$). Take the $4n - 3$ configuration that has the maximum number of perimeter rooks. Because there are $4n - 3$ rooks, there must be a rook in the interior. Since we are assuming it is possible to place $4n - 3$ rooks, this interior rook must have the ability to be "pushed" to the perimeter (for example if it was an interior left rook, the perimeter spot to the left must be open to allow it to work). However, pushing it to the perimeter increases the number of perimeter rooks by 1, and contradicts this being the configuration with the largest number of perimeter rooks, completing the proof.

#1317: *Robert C. Gebhardt, Chester, NJ.*

(a) A continuous differentiable function $g(x) > 0$, $a < x < b$, is revolved about the x-axis to create a surface of revolution with area S, and a volume of revolution V. Find all functions $g(x)$, other than $g(x) = 0$ and $g(x) = 2$, such that the surface area S (in square units) is the same number as the volume V (in cubic units) for all finite choices of a and b (the areas of the discs at each end of the volume are not included).

(b) Let $f(x)$ and $g(x)$ be continuous differentiable functions, $a < x < b$, where $0 < f(x) < g(x)$. Each is revolved about the x-axis to create a volume between them. Find all such functions $f(x)$ and $g(x)$ such that the surfaces' total area S (in square units) and the volume V (in cubic units) are the same number. The washers at each end of the volume are not included.

Solution for part (a) below by Jennifer Johannes, from The College at Brockport, who determined the functional form of $g(x)$ but not the final scale; thus her argument has been replaced by the proposer's argument towards the end.

Proof of (a): Since the volume is

$$V = \pi \int_a^b g(x)^2 dx$$

and the area is

$$S = 2\pi \int_a^b g(x)\sqrt{1+g'^2(x)}dx$$

we are looking for functions $g > 0$ such that

$$\begin{aligned} \pi \int_a^b g(x)^2 dx &= 2\pi \int_a^b g(x)\sqrt{1+g'^2(x)}dx \\ \iff \int_a^b g(x)^2 dx &= 2\int_a^b g(x)\sqrt{1+g'^2(x)}dx \\ \iff \int_a^b \left(g(x)^2 - 2g(x)\sqrt{1+g'^2(x)}\right) dx &= 0 \end{aligned}$$

for every $a \leq b$. This implies that

$$\begin{aligned} g(x)^2 - 2g(x)\sqrt{1+g'^2(x)} &= 0 \\ \iff g(x)^2 &= 2g(x)\sqrt{1+g'^2(x)} \\ \iff g(x) &= 2\sqrt{1+g'^2(x)} \end{aligned}$$

because $g > 0$. This is equivalent to

$$g(x)^2 = 4 + 4g'^2(x).$$

This equation has a constant solution, $g(x) = 2$. *From the proposer:* As $\cosh^2(u) - \sinh^2(u) = 1$, where

$$\cosh(u) = \frac{e^u + e^{-u}}{2} \quad \text{and} \quad \sinh(u) = \frac{e^u - e^{-u}}{2},$$

inspection shows the second solution is $g(x) = 2\cosh(x/2)$.

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