PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2017

STEVEN J. MILLER (EDITOR)

1. Problems: Fall 2017

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by September 15, while solutions for the Fall Issue must arrive by March 15. Congratulations to the Emmanuel College Math Club, which is this issue's winner!

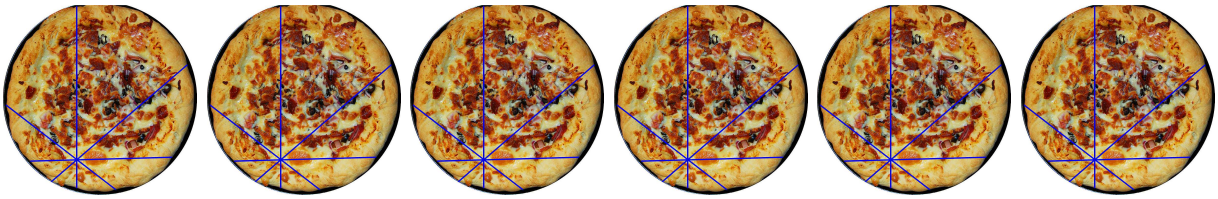

Figure 1. Pizza motivation; can you name the theorem that's represented here?

#1335: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*

Date: September 22, 2017.

The following is from the 1993 Green Chicken Math Competition between Middlebury and Williams Colleges. At State University 7 students registered for American history, 8 students for British history, and 9 students for Chinese history. No student is allowed to take more than one history course at a time. Whenever two students from different classes get together, they decide to drop their current history courses and add the third. Otherwise there are no adds or drops. Is it possible for all students to end up in the same history course?

#1336: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*
The following is from the 1999 Green Chicken Math Competition between Middlebury and Williams Colleges. An integer is powerful if each of its prime factors occurs to the second power or more. Prove or disprove: There are an infinite number of pairs of consecutive powerful numbers.

#1337: *Proposed by Steven J. Miller, Williams College, Williamstown, MA.*
A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2, 3, 4, \ldots, 2017\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus 30 and 1593 are connected by an edge as 3 divides each, but 30 and 49 are not. The coloring number of a graph is the smallest number of colors needed so that each vertex is colored ***and*** if two vertices are connected by an edge, then those two vertices are not colored the same. Prove the coloring number is at least 10. What is the actual value? *This problem was first published in the Newsletter of the European Mathematical Society.*

#1338: *Proposed by Dhruv Desai, University of Illinois at Urbana-Champaign, Champaign, IL.*
Given a set of positive real numbers $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$, prove that it is always possible to find a set of positive real numbers $c_1, d_1, c_2, d_2, \ldots, c_n, d_n$ such that, if at least one of the ratios c_i/d_i is less than $(a_1 + a_2 + \cdots + a_n)/(b_1 + b_2 + \cdots + b_n)$, then we can simultaneously satisfy

- for all $i \in \{1, \ldots, n\}$ we have $a_i/b_i < c_i/d_i$, and
- $(a_1 + a_2 + \cdots + a_n)/(b_1 + b_2 + \cdots + b_n) < (c_1 + c_2 + \cdots + c_n)/(d_1 + d_2 + \cdots + d_n)$.

#1339: *Proposed by Mehtaab Sawhney, University of Pennsylvania, Philadelphia, PA.*
For an integer $n \geq 2$, let $\mathcal{B}$ be a $(2^n - 1)$ by $(n - 1)$ board of 0's and 1's, and let $\mathcal{E}_n(\mathcal{B})$ be the number of contiguous rectangular sub-boards of $\mathcal{B}$ which have an even sum. Give an explicit formula for the minimum of $\mathcal{E}_n(\mathcal{B})$ as we range over all $\mathcal{B}$.

For example, below is one of the possible $\mathcal{B}$ for $n = 3$:

$$\begin{array}{c} 1\ 0\ 0\ 1\ 1\ 0\ 0 \\ 0\ 1\ 0\ 0\ 0\ 1\ 1. \end{array}$$

It has exactly 42 even sub-boards. To be precise there are 8 even sum sub-boards of size 1×1, 1 of size 1×2, 4 of size 2×2, 3 of size 3×2, 1 of size 4×2, 3 of size 5×2, 1 of size 7×2, 6 of size 2×1, 4 of size 3×1, 5 of size 4×1, 4 of size 5×1, and 2 of size 6×1 to give

42 even sum sub-boards in total.

#1340: *Communicated by Steven J. Miller, Williams College*
Zeckendorf proved that if we define the Fibonacci numbers by $F_1 = 1, F_2 = 2$ and $F_{n+2} = F_{n+1} + F_n$ then every integer can be written uniquely as a sum of non-adjacent Fibonacci numbers. We call this the Zeckendorf decomposition; thus $2018 = 1597 + 377 + 34 + 8 + 2$. Prove this claim, and further show that if we write any N as a sum of Fibonacci numbers, no decomposition has fewer summands than the Zeckendorf decomposition.

#1341: *Proposed by Matthew Davis, Williams College*
Consider an infinite one-dimensional board, where we may place checkers at any integer. We initialize the game by placing checkers at all the positive squares, and all their negatives; thus there are checkers only at positions $\pm 1, \pm 4, \pm 9$, and so on. As the game evolves, we may have multiple checkers at the same position (similar to how one may stack checkers in a game to make a king). At each step, you can perform one of several moves.

- You can either add or remove any finite number of checkers.
- Given integers a, k with $k > 1$, you may add a checker at each position ak^n (where n ranges over the non-negative integers).
- If there are integers a, k with $k > 1$ such that for all n there is always at least one checker at position ak^n, then you may remove one checker from each of these positions.

Is it possible to move every checker inward one space in a finite number of moves? In other words, can we reach the state where there are two checkers at 0, and then one each at $\pm 3, \pm 8, \pm 15, \ldots$.

#1342: *Proposed by Ralph Morrison, Williams College*
The following is from the 2016 Green Chicken Math Competition between Middlebury and Williams Colleges. While attending a concert, ***Greenie*** was instructed by the performer to "call her, maybe." Unfortunately ***Greenie*** can't remember the performer's Skype name exactly, but she remembers noticing that it had no repeated digits, was divisible by 3 but not by 6 or by 9, and that it was the largest possible integer satisfying all those properties. What was the number?

GRE Practice #1: *Proposed by Steven Miller, Williams College*
One of the greatest challenges students have with the math GRE subject test is that while they solve a problem, often it is faster to eliminate four wrong answers than find the exact solution (or at least eliminate a few answers, at which point on average it is advantageous to guess). Consider the following old GRE problem (a discussion of the answer is included after the solutions to earlier PME problems). If $a > 0$, what is the value of

$$\int_0^{2a} \int_{-\sqrt{2ay-y^2}}^{0} \sqrt{x^2+y^2}\,dx\,dy.$$

(a) $\frac{16}{9}a^3$ (b) $\frac{32}{9}a^3$ (c) $\frac{\pi}{2}a^2$ (d) $\frac{8\pi}{3}a^2$ (e) $2a^4$.

2. Solutions

In addition to the solutions below, Problems #1323 and #1325 were solved by the Skidmore College Problem Group, but their answers arrived after the Spring 2017 issue went to press.

#1322: *Proposed by Gabriel Prajitura, SUNY Brockport, Brockport, NY 14420*
Construct a sequence of integers $(x_n)_n$ such that the set

$$\left\{x_1,\ \frac{x_1+x_2}{2},\ \frac{x_1+x_2+x_3}{3},\ \ldots\right\}$$

is dense in $\mathbb{R}$.

Solution below by ***Colin Scheibner, St. Olaf Problem Solving Group, St. Olaf College, Northfield, MN*** *(note: this problem solving group also submitted correct solutions to #1323 and #1325 after the previous issue went to press).*

For all $m \in \mathbb{N}_0$, let

$$L_m \ := \ \left\{\frac{k}{2^m} : k \in \mathbb{Z}, -m2^m \le k \le m2^m\right\}.$$

Note that $\cup_{m=0}^{\infty} L_m$ is the set of dyadic fractions, which is dense in $\mathbb{R}$. It suffices to construct a sequence of integers $\{x_n\}$ such that for all $m \in \mathbb{N}_0$, $L_m \subseteq \{y_j\}_{j=1}^{\infty}$, where $y_j = \frac{1}{j}\sum_{n=1}^{j} x_n$. The construction proceeds by induction on n.

For $n = 1$, let $x_1 = 0$. Now for some $N \ge 1$ assume that $\{x_n\}_{n=1}^{N}$ has been constructed and let $M = \max\{m : L_m \subseteq \{y_i\}_{i=1}^{N}\}$. Note that M is well defined since $L_0 \subseteq \{y_1\}$. The proof will be complete if we construct x_{N+1} through x_r (for some $r > N+1$) such that $L_{M+1} \subseteq \{y_j\}_{j=1}^{r}$. Let $x_{N+1} = -\sum_{n=1}^{N} x_n$, and write $k = (M+1)2^{M+1}$. We may choose a set of distinct integers $r_{-k} < r_{-k+1} < \cdots < r_k$ such that 2^{M+1} divides r_i and $r_i > N+1$ for each $i = -k, \ldots, k$. (Note that the r_i are not uniquely determined, but any choice that satisfies the preceding criteria will work.) Define $x_{N+1}, x_{N+2}, \ldots, x_{r_k}$ as follows:

$$x_n \ = \ \begin{cases} \frac{ir_i}{2^{M+1}} & \text{if } n = r_i \text{ for some } i \in \{-k, \ldots, k\} \\ -\frac{ir_i}{2^{M+1}} & \text{if } n = r_i + 1 \text{ for some } i \in \{-k, \ldots, k\} \\ 0 & \text{otherwise.} \end{cases}$$

First note that the definition is unambiguous (i.e., the three cases are disjoint) since $|r_i - r_j| > 1$ whenever $i \ne j$. Note that each x_n is an integer since 2^{M+1} divides each r_i. Also, one may easily check that

$$y_j \ = \ \begin{cases} \frac{i}{2^{M+1}} & \text{if } j = r_i \text{ for some } i \in \{-k, \ldots, k\} \\ 0 & \text{otherwise} \end{cases}$$

whenever $N+1 \le j \le r_k$. Consequently, $L_{M+1} \subseteq \{y_j\}_{j=N+1}^{r_k+1} \subseteq \{y_j\}_{j=1}^{r_k+1}$. Hence the inductive step and the proof are complete.

#1327: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*

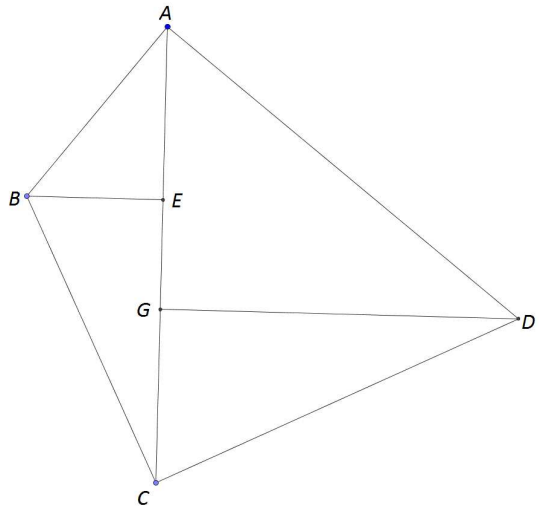

FIGURE 2. Configuration for Problem #1326.

The following is from the 1989 Green Chicken Math Competition between Middlebury and Williams Colleges. The aerial view of a roller-coaster is a perfect circle. Show that there are two diametrically opposed points on the roller-coaster having the same height.

Solution below by ***Emmanuel College Math Club****. Also solved by the* ***Skidmore College Problem Group****.*

For every point x along the roller-coaster, define the function $h(x)$ to be the height of the roller-coaster at x. It is fair to assume that h is continuous, otherwise we have a rather unsafe roller-coaster.

Then, define the function $f(x) = h(x) - h(x')$, where x' is the point diametrically opposed to x. We want to show that some point c makes $f(c) = 0$. Note that f is continuous, as well (if f were discontinuous at x, this would imply h is discontinuous at x or x').

If f is everywhere 0, then we're clearly done, so suppose that some input a makes $f(a) \neq 0$. Without loss of generality, $f(a) > 0$. Let a' be the point diametrically opposed to a. By the definition of f, we have $f(a) < 0$. Since f is continuous, the Intermediate Value Theorem applies and thus if we travel from a to a' clockwise we must pass some point c which satisfies $f(c) = 0$. (Note the IVT is for a function of a real variable; we can use it by writing $a = e^{i\theta_a}$ with $0 \leq \theta_a < 2\pi$.)

#1330: *Proposed by Ioana Mihăilă, Cal Poly Pomona.*
Let $ABCD$ be a quadrilateral with opposite right angles A and C. Let BE and DG be the perpendiculars dropped on AC from B and D respectively (see Figure 2). Show that $AE = GC$.

Solution below by ***Robert O'Connell, Longmeadow High School, Longmeadow, MA****. Also solved by* ***Get Stoked Student Problem Solving Group, Mountain Lakes High School, NJ****,* ***Emmanuel College Math Club****,* ***Nate Vogel, North Central***

***College**, **Arjun Puri, Loyd Templeton and George Zhang, Memphis University School**, and the **Skidmore College Problem Group**.*

Since $\angle DAB$ and $\angle BCD$ are right, then $\angle DAE$ complementary to $\angle BAE$ and $\angle BCG$ complementary to $\angle DCG$. Since BE is perpendicular to AC and DG is perpendicular to AC, then $\angle AEB$, $\angle CEB$, $\angle AGD$, and $\angle CGD$ are right. The acute angles of a right triangle are complementary. Therefore, $\angle BAE$ and $\angle ABE$ are complementary; $\angle DCG$ and $\angle CDG$ are complementary; $\angle DAE$ and $\angle GDA$ are complementary; and $\angle BCG$ and $\angle EBC$ are complementary.

By the Congruent Complements Theorem, $\angle DAE \cong \angle ABE$, $\angle GDA \cong \angle EAB$, $\angle BCG \cong \angle CDG$, and $\angle EBC \cong \angle GCD$. By the AA Triangle Similarity Postulate, triangle DAG is similar to triangle ABE, and triangle BCE is similar to triangle CDG. Thus $AE/DG = BE/AG$ and $GC/BE = DG/CE$, which implies $BE{\cdot}DG = AE{\cdot}AG$ and $BE{\cdot}DG = CE{\cdot}GC$. Therefore,

$$\begin{aligned} AE \cdot AG &= CE \cdot GC \\ AE \cdot (AC - GC) &= (AC - AE) \cdot GC \\ AE \cdot AC - AE \cdot GC &= AC \cdot GC - AE \cdot GC \\ AE \cdot AC &= AC \cdot GC \end{aligned}$$

and thus $AE = GC$.

#1331: *Proposed by Greg Oman, University of Colorado, Colorado Springs.*

A nonempty subset G of $\mathbb{R}$ is an *additive subgroup* of $\mathbb{R}$ provided for any $x, y \in G$, also $x - y \in G$. Now suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is both continuous and injective. Assume further that f preserves additive subgroups of $\mathbb{R}$; that is, if G is an additive subgroup of $\mathbb{R}$, then so is $f[G] := \{f(g)\colon g \in G\}$. Prove that there exists a real number $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$. *Note*: it is well-known that a continuous $f\colon \mathbb{R} \to \mathbb{R}$ for which $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ is necessarily linear. The purpose of this exercise is to show that if one strengthens "continuous" to "continuous and injective", then one can weaken the assumption of additivity to preservation of subgroups.

*Solution below by **Eugen J. Ionascu, Columbus State University**.*

Our proof is based on a few, more or less known, lemmas.

Lemma 2.1. *Every additive subgroup S of $\mathbb{R}$ is either*
(i) cyclic, i.e., $S =< t >:= \{kt : k \in \mathbb{Z}\}$ for some $t \geq 0$, or
(ii) S is dense in $\mathbb{R}$.

Lemma 2.2. *Every $f : \mathbb{R} \to \mathbb{R}$ which is both continuous and injective is strictly increasing or decreasing.*

Lemma 2.3. *Every $f : \mathbb{R} \to \mathbb{R}$ which is both continuous and injective has the property that $f^{-1}(S)$ is dense in $\mathbb{R}$ if S is dense in $\mathbb{R}$.*

Before we prove these lemmas, let's see how the problem can be solved. We consider the cyclic group $G :=< x >= \{kx : k \in \mathbb{Z}\}$ for some $x > 0$. Since $S := f[G]$ is an additive subgroup G of $\mathbb{R}$, by Lemma 2.1, we either have $S =< t >$ $(t > 0)$ or S is dense. In the

second case, by Lemma 2.3, $G = f^{-1}(S)$ is dense in $\mathbb{R}$. Since G is clearly not dense, it remains that $S =< t >$.

By Lemma 2.3, f must be strictly increasing or strictly decreasing. Without loss of generality we may assume it is strictly increasing (otherwise consider $-f$ instead). It is clear that $f(0) = 0$ since $\{0\}$ is the only additive subgroup with only one element. Because x is the smallest positive element of G, then $f(x)$ must be the smallest element of S, i.e. $f(x) = t$. By induction, we see that $f(kx) = kt$ for all $k \in \mathbb{Z}$. So, this can be written as $f(kx) = kf(x)$ for all $k \in \mathbb{Z}$. We can use this argument again but start with $x' := x/m$, for some $m \in \mathbb{N}$. We conclude that $f(mx') = mf(x')$ or $f(x/m) = f(x)/m$. Then

$$f(\frac{k}{m}x) = f(kx') \ = \ kf(x') \ = \ kf(x/m) = \frac{k}{m}f(x), \quad k \in \mathbb{Z}.$$

Since m was arbitrary, taking $x = 1$ and setting $a = f(1)$ we obtain $f(q) = qa$ for all $q \in \mathbb{Q}$. Becasue f is continuous, this equality should be taking place for all real q, hence f must be linear.

Proof of Lemma 2.1. If $S = \{0\}$ then clearly, the first case applies. So, we may assume that $S \neq \{0\}$. Hence, we may define

$$t := \inf\{s : s > 0, s \in S\}.$$

There are two possiblities. Either $t = 0$ or $t > 0$. If $t = 0$, then let us show that S is dense. Indeed, for every fixed $x \in \mathbb{R}$, and $\epsilon > 0$, consider $s \in S$ so that $0 < s \leq \epsilon$. Then, let k be the greatest integer such that $ks \leq x$, i.e., $k = \lfloor x/s \rfloor$. This implies $ks \leq x < (k+1)s$ or $0 \leq x - ks < s \leq \epsilon$. Since $ks \in S$, it follows that S is dense in $\mathbb{R}$.

If $t > 0$, then every element in S must be of the form ks for some integer k. By way of contradiction, suppose this were not true. Then, there exists $x \in S$ not of the form ks. Let $\ell = \lfloor x/t \rfloor$ and observe that $\ell t < x < (\ell + 1)t$. This means that $t' := x - \ell t$ has the properties: $t' > 0$, $t' \in S$ and $t' < t$. This contradicts the definition of t. It remains that $S =< t >$. □

Proof of Lemma 2.2. Let us assume by way of contradiction that f is neither strictly increasing nor strictly decreasing. Then there exits x, y, z such that $x < y < z$ and so that $f(x) < f(y)$ and $f(z) < f(y)$ or the other way around, i.e, $f(x) > f(y)$ and $f(z) > f(y)$. Using the Intermediate Value Theorem (IVT), we can find $c_1 \in (x, y)$ and $c_2 \in (y, z)$ so that $f(c_1) = f(c_2) = \xi$ with ξ strictly between $f(x)$ and $f(y)$ and also between $f(z)$ and $f(y)$. This contradicts the assumption that f is injective. □

Proof of Lemma 2.3. Suppose by way of contradiction that $f^{-1}(S)$ is not dense. Then there exists and interval $I = (a, b)$ included in the complement of $f^{-1}(S)$. Then $f(I)$ is included in the complement of S. By the IVT, $f(I)$ is an interval and since f is injective $f(I)$ is not a point. This contradicts the assumption that S is dense in $\mathbb{R}$. □

#1333: Proposed by Steven R. Conrad, Math League.
Three circles are all externally tangent, and the lengths of their radii are 1, 4/9, and r. Find all values of r (if any) for which a fourth circle can surround the first three so they are all internally tangent to it.

Solution below by **Emmanuel College Math Club**.

We invoke a theorem from geometry about this situation to establish an inequality involving r. Solving for r will show that it must satisfy

$$0 \;<\; r \;<\; \frac{1}{63}\left(\sqrt{2032}-32\right) \;\approx\; 0.20758\ldots.$$

The `MathWorld` *page for* ***Soddy Circles*** *cites a paper by Coxeter in the American Mathematical Monthly from 1968. This paper presents a proof about the situation stated in this problem, which is in turn due to Frederick Soddy (Nature, 1936). The result is that, given three mutually externally tangent circles and a fourth circle that is mutually tangent to the other three, we have*

$$2(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2+\epsilon_4^2) \;=\; (\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)^2$$

where $\epsilon_i = 1/r_i$ is the curvature of circle i (and r_i is its radius), and circle $i=4$ is the fourth one tangent to the other three. Solving for r_4 via the quadratic formula shows that there are two solutions, and they correspond to the fact that the fourth circle could be in the region between the other three, or outside them (so that they're internally tangent to it, as desired here). The solution for r_4 corresponding to that desired situation is:

$$r_4 \;=\; \frac{r_1r_2r_3}{r_1r_2+r_1r_3+r_2r_3-2\sqrt{r_1r_2r_3(r_1+r_2+r_3)}}.$$

In this problem, we have $r_1=1$ and $r_2=4/9$ and $r_3=r$. Substituting these values and simplifying yields

$$r_4 \;=\; \frac{4r/9}{\frac{4r}{9}(r+1)-2\sqrt{\frac{4r}{9}\left(\frac{13}{9}+r\right)}}.$$

Now, $r_4>0$ exists if and only if the denominator is strictly positive:

$$\frac{4r}{9}(r+1) \;-\; 2\sqrt{\frac{4r}{9}\left(\frac{13}{9}+r\right)} \;>\; 0 \iff \frac{4r}{9}(r+1) \;>\; 2\sqrt{\frac{4r}{9}\left(\frac{13}{9}+r\right)} \;>\; 0.$$

With all terms positive, we may square both sides

$$\frac{16}{81}(r+1)^2+\frac{8}{9}r(r+1)+r^2 \;>\; 4\left(\frac{4r}{9}\left(\frac{13}{9}+r\right)\right)$$

and then simplify

$$r^2+\frac{144}{81}r+\frac{16}{81} \;>\; \frac{16}{9}r^2+\frac{208}{81}r.$$

Bringing all terms to one side and multiplying through by 81 yields

$$0 \;>\; 63r^2+64r-16.$$

Completing the square yields

$$0 \;>\; \frac{1}{63}\left((63r+32)^2-2032\right),$$

which is true if and only if

$$0<(63r+32)^2 \;<\; 2032 \iff 63r+32 \;<\; \sqrt{2032} \iff r \;<\; \frac{\sqrt{2032}-32}{63}.$$

GRE Practice #1: Proposed by Steven Miller, Williams College

One of the greatest challenges students have with the math GRE subject test is that while they solve a problem, often it is faster to eliminate four wrong answers than find the exact solution (or at least eliminate a few answers, at which point on average it is advantageous to guess). Consider the following old GRE problem (a discussion of the answer is included after the solutions to earlier PME problems). If $a > 0$, what is the value of

$$\int_0^{2a} \int_{-\sqrt{2ay-y^2}}^{0} \sqrt{x^2+y^2}dxdy.$$

(a) $\frac{16}{9}a^3$ *(b)* $\frac{32}{9}a^3$ *(c)* $\frac{\pi}{2}a^2$ *(d)* $\frac{8\pi}{3}a^2$ *(e)* $2a^4$.

Solution: *This problem allows me to introduce one of my favorite techniques. I like to use the physics perspesctive and talk about dimensional analysis. Consider for example the integral* $\int_0^3 x^3 e^{ax}dx$. *If we imagine x is in meters then a must have units of* meters^{-1}, *as the argument of the exponential must be unitless (to see this consider the Taylor series* $e^u = 1+u+u^2/2!+u^3/3!+\cdots$; *if u had units then we would be adding quantities of different dimensions). Thus as the integrand has units of* meters^4 *the integral must have units of* meters^4. *We integrate out x, and thus the answer will be a polynomial in $1/a$ up to degree 4 (we have to remember that the bounds of integration of 0 and 3 have units of meters, but we expect the leading term to be $1/a^4$).*

Let's apply this logic to our double integral

$$\int_0^{2a} \int_{-\sqrt{2ay-y^2}}^{0} \sqrt{x^2+y^2}dxdy.$$

If we let x and y be in meters then the integrand has units of meters^3. *We must have a in meters so that $2a$ has the same dimension as y. Thus we expect the answer to look like a multiple of a^3, which eliminates all but (a) and (b). Of course, similar to our analysis of the exponential integral, we should be careful as perhaps a constant times a^2 really has units of* meters^3. *To show that it really has to be one of these answers we just need to explore what happens as $a \to \infty$ and see that the integral grows like a^2 and not a lower power (we will see this shortly).*

Even if we cannot simplify further, we have made great progress and have reduced it to essentially a 50-50 chance. We can do better, however. Instead of trying to calculate the integral exactly we can estimate. We just need to find a lower bound greater than (a) to prove it is (b), or an upper bound smaller than (b) to prove it is (a). A simple bound is to use $\sqrt{x^2+y^2} \le \sqrt{x^2}+\sqrt{y^2} = |x|+y$ *(we have to be careful and remember that x is negative in the region of integration). Notice that* $|x| \le \sqrt{x^2+y^2}$, *so this would give a lower bound and will prove it grows at least like a^3.*

Integrating x over the region gives

$$\int_0^{2a} \int_{-\sqrt{2ay-y^2}}^{0} |x|dxdy \;=\; \int_0^{2a} \frac{2ay-y^2}{2}dy \;=\; \frac{2a^3}{3},$$

while integrating y gives

$$\int_0^{2a}\int_{-\sqrt{2ay-y^2}}^{0} y\,dx\,dy \;=\; \int_0^{2a} y\sqrt{2ay-y^2}\,dy \;\le\; \int_0^{2a}\sqrt{2a}y^{3/2} \;=\; \frac{2}{5}\sqrt{2a}(2a)^{5/2} \;=\; \frac{4\sqrt{2}}{5}a^3.$$

Combining, we see the double integral is at most $\frac{4\sqrt{2}}{5}+\frac{2}{3}<3$; as this is less than $32/9$, we see that (b) is too large and the answer should be (a).

Depending on how much time we are willing to spend, we can either remove all but one answer, or all but two. Notice we are able to do these relatively quickly, significantly faster than doing the actual double integral. It's important to remember that for the purposes of this test it does not matter how *you reach your answer, only what answer you reach. Thus if you can eliminate four answers faster than finding one.... Also, we were fortunate here that crude estimation sufficed to eliminate the last option; we didn't have to evaluate the integral too well.*

E-mail address: sjm1@williams.edu

Associate Professor of Mathematics, Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267