PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2017

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2017

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged.

We have made some changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to \$100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. We are happy to report that three schools qualified: Andrews University, Cal Poly Pomona, The Episcopal Academy, and North Central College; the randomly selected one was North Central College: Congrats!

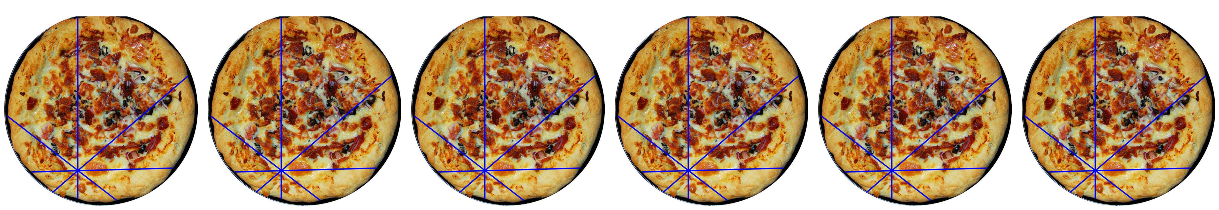

Figure 1. Pizza motivation; can you name the theorem that's represented here?

#1327: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*

Date: April 11, 2017.

The following is from the 1989 Green Chicken Math Competition between Middlebury and Williams Colleges. The aerial view of a roller-coaster is a perfect circle. Show that there are two diametrically opposed points on the roller-coaster having the same height.

#1328: *Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.*

Let a sequence $\{a_n\}$ satisfy

$$a_{n+1} \;=\; \frac{(2n-1)a_n - 9(n-2)a_{n-1}}{n+1}$$

for $n \geq 1$ and set $a_1 = 1$ and $a_2 = -1$. Prove that a_n is integral if $n \in \mathbb{Z}^+$. *Hint: Note the similarity of the recursion to those of Motzkin numbers, Delannoy numbers, and super-Catalan numbers, though "combinatorially" it is different.*

#1329: *Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.*

There is a beautiful closed form expression for the number of ways to tile a regular hexagon with edge length of n with diamonds of side length 1 and angles 60° and 120°; it is

$$\prod_{i=0}^{n-1} \frac{(i)!(i+2n)!}{[(i+n)!]^2}.$$

This formula is a special case of MacMahon's formula which considers the more general problem of plane partitions; see the sequence A008793 in the Online Encyclopedia of Integer Sequences (OEIS) (https://oeis.org/A008793) or https://oeis.org/wiki/Plane_partitions for further information. MacMahon's formula is traditionally proved using a tricky generating function argument, and even our special case is no simpler (see https://aquazorcarson.wordpress.com/2011/02/25/ for an excellent presentation of such an argument). Prove directly that this quantity is in fact an integer without resorting to the combinatorial interpretation. *Hint: Use Legendre's Formula, which says the largest power of a prime p dividing an integer m is $\sum_{\ell=1}^{\infty} \lfloor m/p^\ell \rfloor$, where $\lfloor x \rfloor$ is the greatest integer at most x.*

#1330: *Proposed by Ioana Mihăilă, Cal Poly Pomona.*

Let $ABCD$ be a quadrilateral with opposite right angles A and C. Let BE and DG be the perpendiculars dropped on AC from B and D respectively (see Figure 2). Show that $AE = GC$.

#1331: *Proposed by Greg Oman, University of Colorado, Colorado Springs.*

A nonempty subset G of $\mathbb{R}$ is an *additive subgroup* of $\mathbb{R}$ provided for any $x, y \in G$, also $x - y \in G$. Now suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is both continuous and injective. Assume further that f preserves additive subgroups of $\mathbb{R}$; that is, if G is an additive subgroup of $\mathbb{R}$, then so is $f[G] := \{f(g)\colon g \in G\}$. Prove that there exists a real number $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$. *Note*: it is well-known that a continuous $f\colon \mathbb{R} \to \mathbb{R}$ for which $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ is necessarily linear. The purpose of this exercise is to show that if one strengthens "continuous" to "continuous and injective", then one can weaken the assumption of additivity to preservation of subgroups.

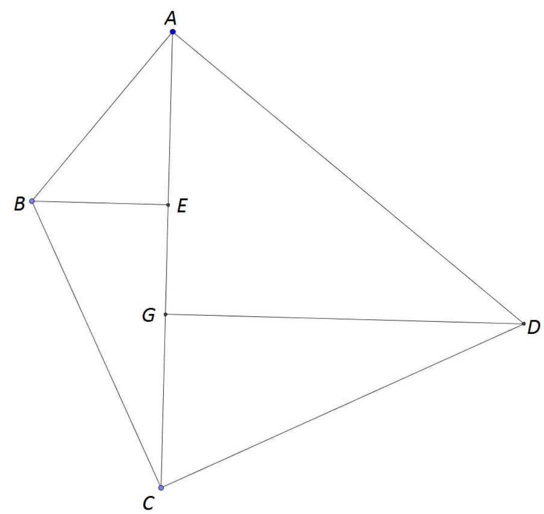

FIGURE 2. Configuration for Problem #1326.

#1332: *Proposed by Greg Oman, University of Colorado, Colorado Springs.*

Find all fields F and F-vector spaces V with the property that any two bases for V have nonempty intersection. *Note: the text below is meant to be a brief introduction to the concepts of fields, vector spaces and bases; for more information see any linear algebra book, or search on Google or Wikipedia. One reason we have chosen to include this problem is to encourage readers who have not seen these concepts to explore them.*

The operations of addition and multiplication on the set $\mathbb{R}$ of real numbers enjoy many familiar properties. For example, both operations are commutative and associative, and multiplication distributes over addition. A mathematical structure with operations (denoted by $+$ and $\cdot$) which enjoy similar properties is called a *field*. One can add two ordered n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of real numbers in a natural way by defining $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n)$. We can "multiply" a real number r by $(x_1, \ldots, x_n)$ to get another ordered n-tuple of real numbers by defining $r \cdot (x_1, \ldots, x_n) := (rx_1, \ldots, rx_n)$. One can verify that again, many familiar algebraic properties are enjoyed by these operations. For instance, $+$ is commutative and $\cdot$ distributes over addition. A mathematical structure with operations $+$ and a scalar multiplication $\cdot$ over a field F which enjoy similar properties is called a *vector space* over F. Consider now the vector space $\mathbb{R}^2$ of ordered pairs of real numbers with operations defined above. Observe that if $(a, b) \in \mathbb{R}^2$ is arbitrary, then $(a, b) = a(1, 0) + b(0, 1)$. Thus every ordered pair of real numbers can be expressed as a *linear combination* of $(1, 0)$ and $(0, 1)$. More generally, if V is a vector space over a field F, then a subset S of V is said to *span* V if every element of V is a finite linear combination of elements of S; one also says that S is a *spanning subset* of V. A spanning subset S of V with no proper spanning subsets is called a *basis* of V.

#1333: *Proposed by Steven R. Conrad, Math League.*

Three circles are all externally tangent, and the lengths of their radii are 1, 4/9 and r. Find all values of r (if any) for which a fourth circle can surround the first three so they are all internally tangent to it.

2. Solutions

#1312: *Proposed by Steven J. Miller, Department of Mathematics and Statistics, Williams College, and Stand Wagon, Department of Mathematics, Statistics, and Computer Science, Macalester College.*

Larry Bird and Magic Johnson are playing a game of basketball; they alternate shooting with Bird going first, and the first to make a basket wins. Assume Bird always makes a shot with probability p_B and Magic with probability p_M, where p_B and p_M are independent uniform random variables. (This means the probability each of them is in $[a, b] \subset [0, 1]$ is $b - a$, and knowledge of the value of p_B gives no information on the value of p_M.)

(1) What is the probability Bird wins the game?
(2) What is the probability that, when they play, Bird has as good or greater chance of winning than Magic?

Solution below by ***Nate Vogel and David Schmitz, North Central College Math Department****.*

To find the probability that Bird wins the game, we first need to find the probability that Bird wins on his n^{th} shot. So, for any n shot attempt, the probability that Bird wins on shot n, denoted here B_w, is

$$P_n(B_w) \;=\; P_B\left((1 - P_B)\,(1 - P_M)\right)^{n-1}.$$

Bird's chance of winning the game is

$$P(B_w) \;=\; \sum_{n=1}^{\infty} P_n(B_w) \;=\; \sum_{n=1}^{\infty} P_B\left((1 - P_B)\,(1 - P_M)\right)^{n-1}.$$

Notice that this infinite sum takes the form of a geometric series, with

$$\begin{aligned} r &= (1 - P_B)\,(1 - P_M)\,, \\ a &= P_B. \end{aligned}$$

We can be assured that r is between 0 and 1, as it is a product of probabilities.
The sum of a geometric series takes the form of

$$\frac{a}{1-r} \;=\; \frac{P_B}{1 - (1 - P_B)(1 - P_M)} \;=\; P(B_w).$$

This is Bird's chance of winning in general terms.

Next, we compute the exact value of Bird's chance of winning by integrating this expression over the square of $[0, 1] \times [0, 1]$. Letting $P_M = y$ and $P_B = x$, we set up our desired integral as follows:

$$\int_0^1 \int_0^1 \frac{x}{1 - (1 - x)(1 - y)} dydx.$$

To simplify our process, let us first solve the inner integral. We use u-substitution, letting $u = 1-(1-x)(1-y)$, and $du = (1-x)dy$.

$$\begin{aligned}
\int_0^1 \frac{x}{1-(1-x)(1-y)}dy &= \int_x^1 \frac{x}{(1-x)(u)}du \\
&= \frac{x}{1-x}\ln u\Big|_x^1 \\
&= 0 - \frac{x\ln x}{1-x} \\
&= -\frac{x\ln x}{1-x}.
\end{aligned}$$

Now, we want to simplify our outer integral into a more workable difference of integrals, where we let $z = 1-x$.

$$\begin{aligned}
-\int_0^1 \frac{x\ln x}{1-x}dx &= \int_1^0 \frac{(1-z)\ln(1-z)}{z}dz \\
&= \int_0^1 \frac{(z-1)\ln(1-z)}{z}dz \\
&= \int_0^1 \frac{z\ln(1-z)}{z} - \frac{\ln(1-z)}{z}dz \\
&= \int_0^1 \ln(1-z)dz - \int_0^1 \frac{\ln(1-z)}{z}dz.
\end{aligned}$$

Solving for our first integral in our integral difference by using w-substitution, where we have let $w = 1-z$, $dw = -dz$,

$$\begin{aligned}
\int_0^1 \ln(1-z)dz &= -\int_1^0 \ln w dw \\
&= \int_0^1 \ln w dw \\
&= \lim_{t\to 0^+}\int_t^1 \ln w dw \\
&= \lim_{t\to 0^+}\left(w\ln w - w\Big|_t^1\right) \\
&= \lim_{t\to 0^+}\left(-1-(t\ln t - t)\right) \\
&= -1
\end{aligned}$$

Now, solving for the second integral in our integral difference,

$$\begin{aligned}
\int_0^1 \frac{\ln(1-z)}{z}dz &= -\mathrm{Li}_2(1) \\
&= -\left(\frac{-\pi^2}{6}\right),
\end{aligned}$$

where we have applied one of the integral equivalencies of the dilogarithmic function (see Wolfram MathWorld's article for reference - http://mathworld.wolfram.com/Dilogarithm.html).

Finally, subtracting these two results gets us Bird's probability of winning,

$$-1-\left(-\frac{\pi^2}{6}\right) \;=\; \frac{\pi^2}{6}-1.$$

For the second part, let us denote in general terms the probability that Bird has as good or greater a chance of winning than Magic as

$$P(P(B_w) \geq P(M_w)) \;=\; P(P(B_w) - P(M_w) \geq 0).$$

Notice also that

$$P(M_w) \;=\; \frac{P_M(1-P_B)}{1-(1-P_B)(1-P_M)},$$

where the same method for finding Bird's chance of winning the game can be used to find Magic's. Again, letting $P_M = y$ and $P_B = x$, we can simplify our probability

$$\begin{aligned}
P(P(B_w) - P(M_w) \geq 0) &= P(P_B - P_M(1-P_B) \geq 0) \\
&= P(x - y(1-x) \geq 0) \\
&= P(x - y + xy \geq 0) \\
&= P(y \leq x + xy) \\
&= P(y - xy \leq x) \\
&= P\left(y \leq \frac{x}{1-x}\right).
\end{aligned}$$

Now we can use a double integral to find the area under the curve $y = \frac{x}{1-x}$ (see Figure 3).

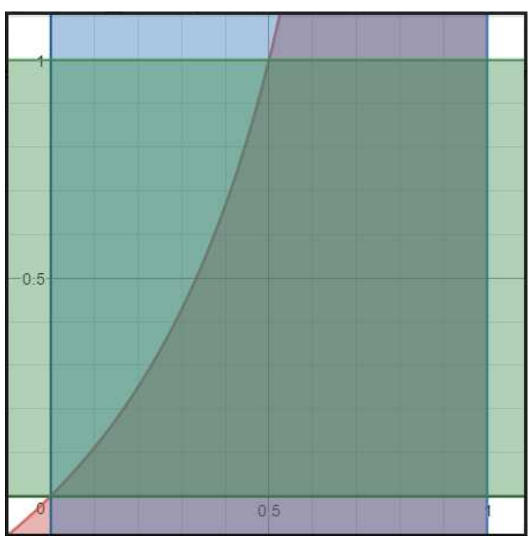

Figure 3. Area under $y = x/(1-x)$.

We can split the area that we are trying to find into two parts. The right section is a rectangle of area $\frac{1}{2}$. The remaining area to be calculated can be found using the following integral:

$$\int_0^{\frac{1}{2}} \int_0^{\frac{x}{1-x}} dydx,$$

where our bounds are found using our simplified probability inequality. To solve this integral, we use u-substitution, letting $u = 1 - x$, $du = -dx$, and $x = 1 - u$. Substituting these in and simplifying, we have

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{u-1}{u} du &= \int_0^{\frac{1}{2}} 1 - \frac{1}{u} du \\ &= u - \ln(u) \Big|_1^{\frac{1}{2}} \\ &= \ln(2) - \frac{1}{2} \end{aligned}$$

Finally, to find the total area, and our desired probability, we simply add $\frac{1}{2}$ to our answer, which gives us $\ln(2)$ as our final answer.

#1318: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*

The following is from the 2009 Green Chicken Math Competition between Middlebury and Williams Colleges. Evaluate

$$\sum_{n=1}^{\infty} \sin(2\pi/3^n) \sin(\pi/3^n).$$

Solution below by ***Robert C. Gebhardt, Chester, NJ.*** *Also solved by* ***the Episcopal Academy Problem Solvers (Shreya Dalal, Laura Lewis, Stephana Lim, Abhay Malik, Sameer Saxena, Jake Viscusi (students) and Tom Goebeler), The Episcopal Academy, Newtown Square, PA****,* ***John Kampmeyer, Elizabethtown College****.*

As

$$\sin\alpha \sin\beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2},$$

we have

$$\sin\frac{2\pi}{3^n} \sin\frac{\pi}{3^n} = \frac{1}{2}\left(\cos\frac{\pi}{3^n} - \cos\frac{3\pi}{3^n}\right).$$

Thus our sum equals

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{2}\left(\cos\frac{\pi}{3^n} - \cos\frac{3\pi}{3^n}\right) &= \frac{1}{2}\left(\cos\frac{\pi}{3} - \cos\pi\right) + \frac{1}{2}\left(\cos\frac{\pi}{9} - \cos\frac{\pi}{3}\right) \\ &\quad + \frac{1}{2}\left(\cos\frac{\pi}{27} - \cos\frac{\pi}{9}\right) + \frac{1}{2}\left(\cos\frac{\pi}{81} - \cos\frac{\pi}{27}\right) + \cdots . \end{aligned}$$

Notice that pairs of terms cancel, leaving only

$$\frac{1}{2}(-\cos\pi) + \frac{1}{2}\cos 0 \;=\; 1.$$

#1319: *Proposed by Mehtaab Sawhney, Commack High School, 6 Roanoke Ct., Commack, NY 11725.*

A classic linear algebra problem is to calculate the determinant of the symmetric Pascal Matrix, P, which is the $n \times n$ matrix whose $(i,j)^{\text{th}}$ entry is $P_{i,j} = \binom{i+j}{j}$. Notice that the Pascal Matrix satisfies $P_{i,j} = P_{i-1,j} + P_{i,j-1}$ for i and j greater than 1, which is equivalent to Pascal's Identity. Building on this identity, we can consider the more general family of matrices $A(n)$ such that $A(n)$ is an $n \times n$ matrix whose entries satisfy

$$A(n)_{i,j} \;=\; \begin{cases} 1 & i = 1 \text{ and/or } j = 1 \\ A(n)_{i-1,j} + A(n)_{i,j-1} + kA(n)_{i-1,j-1} & \text{otherwise;} \end{cases}$$

notice that the symmetric Pascal Matrix is simply the case $k = 0$. Determine $\det(A(n))$ as a function of k and n.

Hint: It is known that the Pascal Matrix has an LU factorization as a consequence of Vandermonde's Identity. One approach (although not the easiest) for this problem is to show that $A(n)$ has an LU factorization and then use this factorization to compute the determinant.

Solution below by ***Steve Edwards, Kennesaw State University, Marietta, GA****.*

Since the entries of the matrix are independent of n, we will write $a_{i,j}$ for $A(n)_{i,j}$. We first show that for $m \geq 2$,

$$\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} a_{m-i,n} \;=\; \binom{n-1}{m-1}(k+1)^{m-1}. \qquad (1)$$

Note that $a_{2,n} = (n-1)k + n$ for all positive integers n. For $m = 2$, we have

$$\sum_{i=0}^{1} (-1)^i \binom{1}{i} a_{2-i,n} \;=\; a_{2,n} - a_{1,n} \;=\; [(n-1)k+n]-1 \;=\; (n-1)(k+1) \;=\; \binom{n-1}{1}(k+1)^1.$$

We proceed by strong induction on m, with (1) as our inductive hypothesis. Then for $n = 1$, the sum becomes an alternating sum of a row from Pascal's triangle:

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{m+1-i,1} \;=\; \sum_{i=0}^{m} (-1)^i \binom{m}{i} \;=\; 0 \;=\; \binom{0}{m}(k+1)^m.$$

We next proceed by induction on n, i.e., assume that for some positive integer n,

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{m+1-i,n} \;=\; \binom{n-1}{m}(k+1)^m.$$

Then

$$\begin{aligned}
&\sum_{i=0}^{m}(-1)^i\binom{m}{i}a_{m+1-i,n+1} \qquad (2)\\
= \quad &\left[\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}a_{m+1-i,n+1}\right]+(-1)^m a_{1,n+1}\\
= \quad &\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}a_{m-i,n+1}+\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}a_{m+1-i,n}+k\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}a_{m-i,n}+(-1)^m a_{1,n}\\
= \quad &\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}(a_{m-i,n+1}+ka_{m-i,n})+\sum_{i=0}^{m}(-1)^i\binom{m}{i}a_{m+1-i,n}.
\end{aligned}$$

Note next that $a_{1,n}=\binom{1}{1}a_{1,n}=1=\binom{n-1}{0}$, and $a_{2,n}-\binom{2}{1}a_{1,n}=(\binom{1}{0}a_{2,n}-\binom{1}{1}a_{1,n})-a_{1,n}=\binom{n-1}{1}(k+1)-\binom{n-1}{0}$. It follows more generally from Pascal's Identity, the inductive hypothesis on m, and induction on n that

$$\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}a_{m-i,n} \;=\; \sum_{i=0}^{m-1}(-1)^i\binom{n-1}{m-1-i}(k+1)^{m-1-i}.$$

Thus

$$\begin{aligned}
&\sum_{i=0}^{m-1}(-1)^i\binom{m}{i}(a_{m-i,n+1}+ka_{m-i,n})\\
= \quad &\sum_{i=0}^{m-1}(-1)^i\binom{n}{m-1-i}(k+1)^{m-1-i}+k\sum_{i=0}^{m-1}(-1)^i\binom{n-1}{m-1-i}(k+1)^{m-1-i}.
\end{aligned}$$

Now let $b=k+1$, which makes this last expression

$$\begin{aligned}
&\sum_{i=0}^{m-1}(-1)^i\binom{n}{m-1-i}b^{m-1-i}+(b-1)\sum_{i=0}^{m-1}(-1)^i\binom{n-1}{m-1-i}b^{m-1-i}\\
= \quad &\sum_{i=0}^{m-1}(-1)^i\binom{n}{m-1-i}b^{m-1-i}+\sum_{i=0}^{m-1}(-1)^i\binom{n-1}{m-1-i}b^{m-i}-\sum_{i=0}^{m-1}(-1)^i\binom{n-1}{m-1-i}b^{m-1-i}.
\end{aligned}$$

Now use Pascal's Identity on the first and third sum to get

$$\sum_{i=0}^{m-2}(-1)^i\binom{n-1}{m-2-i}b^{m-1-i}+\sum_{i=0}^{m-1}(-1)^i\binom{n-1}{m-1-i}b^{m-i}.$$

Re-index the first sum to get

$$\sum_{i=1}^{m-1}(-1)^{i+1}\binom{n-1}{m-1-i}b^{m-i}+\sum_{i=0}^{m-1}(-1)^i\binom{n-1}{m-1-i}b^{m-i} \;=\; \binom{n-1}{m-1}b^m \;=\; \binom{n-1}{m-1}(k+1)^m.$$

Finally, this and the inductive hypothesis on n make (2) equal to

$$\binom{n-1}{m-1}(k+1)^m+\binom{n-1}{m}(k+1)^m \;=\; \binom{n}{m}(k+1)^m.$$

This completes the inductions. Because of (1), when $m = n$,

$$\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}a_{n-i,n} \;=\; \binom{n-1}{n-1}(k+1)^{n-1} \;=\; (k+1)^{n-1},$$

and for $m > n$,

$$\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}a_{m-i,n} \;=\; \binom{n-1}{m-1}(k+1)^{m-1} \;=\; 0.$$

Now let R_j represent the j^{th} row of $A(n)$. Then if R_j is replaced by

$$\sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}R_{j-i}$$

in the order $j = n, n-1, \ldots, 2$, the result is an upper triangular matrix with diagonal entries $1, (k+1), (k+1)^2, \ldots, (k+1)^{n-1}$. This shows that $\det(A(n)) = (k+1)^{1+2+\cdots+(n-1)}$, or $\det(A(n)) = (k+1)^{\frac{n(n-1)}{2}}$. Finally, we note that the Pascal Matrix is the case $k = 0$, so this shows that the determinant of the Pascal Matrix is 1.

We also include the solution by the proposer. Let $|M|$ denote the determinant of a matrix M. Define a new matrix $B(n)$ by taking the $n-1^{\text{st}}$ column of $A(n)$ and subtracting it from the n^{th} column of $A(n)$, $n-2^{\text{nd}}$ column of $A(n)$ and subtracting from the $n-1^{\text{st}}$ column of $A(n)$, and so on until the taking the 1^{st} column of $A(n)$ and subtract it from the 2^{nd} column of $A(n)$. Since elementary row operation preserve the determinant it follows that $|A(n)| = |B(n)|$. Furthermore it also follows that

$$B(n)_{i,j} = \begin{cases} A(n)_{i,j} & j = 1 \\ A(n)_{i,j} - A(n)_{i-1,j} & \text{otherwise,} \end{cases}$$

and now we define a new matrix $C(n)$. Create $C(n)$ by taking the $n-1^{st}$ row of $B(n)$ and subtracting it from the n^{th} row of $B(n)$, taking $n-2^{\text{nd}}$ row and subtracting it from the $n-1^{\text{st}}$ row , and so on until taking the 1^{st} row of $C(n)$ and subtract in from the 2^{nd} row of $C(n)$. As before it follows that $|B(n)| = |C(n)|$ and it follows that

$$C(n)_{i,j} = \begin{cases} A(n)_{i,j} & i = 1 \text{ and } j = 1 \\ A(n)_{i,j} - A(n)_{i,j-1} & i = 1 \text{ and } j \neq 1 \\ A(n)_{i,j} - A(n)_{i-1,j} & j = 1 \text{ and } i \neq 1 \\ A(n)_{i,j} - A(n)_{i-1,j} - A(n)_{i-1,j} + A(n)_{i-1,j-1} & \text{otherwise.} \end{cases}$$

Using the conditions of the original problem it is possible to simplify $C(n)$. Notice that $A(n)_{i,j} - A(n)_{i-1,j} - A(n)_{i-1,j} + A(n)_{i-1,j-1} = kA(n)_{i-1,j-1} + A(n)_{i-1,j-1} = (k+1)A(n)_{i-1,j-1}$ for $i \neq 1$ and $j \neq 1$ and therefore it follows that

$$C(n)_{i,j} = \begin{cases} 1 & i = 1 \text{ and } j = 1 \\ 0 & i = 1 \text{ and } j \neq 1 \\ 0 & j = 1 \text{ and } i \neq 1 \\ (k+1)A(n)_{i-1,j-1} & \text{otherwise,} \end{cases}$$

and expanding along the first row it trivially follows that $|C(n)| = |B(n)| = |A(n)| = (k+1)^{n-1}|A(n-1)|$. (Separate out a factor of $k+1$ from all columns except the first to obtain the last equality). Since $|A(1)| = 1$, it follows from trivial induction that $|A(n)| = (k+1)^{\frac{n(n-1)}{2}}$ and the result follows.

#1323: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*
The following is from the 1997 Green Chicken Math Competition between Middlebury and Williams Colleges. Does any row of Pascal's triangle have three consecutive entries that are in the ratio 1:2:3?

Solution below by ***Mitchell Eithun, Ripon College.*** *Also solved by* ***Robert C. Gebhardt, Chester, NJ****,* ***Hongwei Chen, Christopher Newport University, VA****,* ***Jeremiah Bartz, University of North Dakota, ND****,* ***Ioana Mihaila, Cal Poly Pomona****,* ***Ashland University Undergraduate Problem Solving Group, Ashland, OH****,* ***Tommy Goebeler and Sameer Saxena, The Episcopal Academy, Newtown Square, PA****,* ***The Pi Mu Epsilon chapter at Andrews University, Berrien Springs, MI****.*
Recall that the three consecutive entries in row n of Pascal's Triangle starting at column p are the binomial coefficients $\binom{n}{p}$ and $\binom{n}{p+1}$ and $\binom{n}{p+2}$. If $\binom{n}{p}$ and $\binom{n}{p+1}$ are in a 1:2 ratio, then

$$\frac{1}{2} = \frac{\binom{n}{p}}{\binom{n}{p+1}} = \frac{\dfrac{n!}{p!(n-p)!}}{\dfrac{n!}{(p+1)!(n-p-1)!}} = \frac{p+1}{n-p}.$$

Using cross multiplication, $2(p+1) = n-p$ or $n = 3p+2$. Similarly, if $\binom{n}{p+1}$ and $\binom{n}{p+2}$ are in a 2:3 ratio, then

$$\frac{2}{3} = \frac{\binom{n}{p+1}}{\binom{n}{p+2}} = \frac{\dfrac{n!}{(p+1)!(n-p-1)!}}{\dfrac{n!}{(p+2)!(n-p-2)!}} = \frac{p+2}{n-p-1}.$$

Hence, $3(p+2) \;=\; 2(n-p-1)$. Using $n = 3p+2$, we have

$$\begin{aligned} 3(p+2) &= 2((3p+2)-p-1) \\ 3p+6 &= 4p+2 \\ p &= 4. \end{aligned}$$

We find $n = 3(4)+2 = 14$. Therefore, in the 14^{th} row of Pacsal's triangle, entries 4, 5 and 6 form a 1:2:3 ratio (the entries are 1001, 2002 and 3003).

#1325: *Proposed by Matthew McMullen, Otterbein U., Westerville, OH.*
An equable triangle is a triangle whose area is numerically equal to its perimeter. Find infinitely many (or, better yet, *all*) right, equable triangles with rational side lengths.

Solution below by ***Satyanand Singh, New York City College of Technology of the CUNY, Brooklyn, NY.*** *Also solved by* ***Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX***, ***Ioana Mihaila, Cal Poly Pomona***, ***Mike Lucia, North Central College, Naperville, Illinois***, ***The Pi Mu Epsilon chapter at Andrews University, Berrien Springs, MI***, ***Ashland University Undergraduate Problem Solving Group, Ashland, OH***, ***Hope Miedema, Kaitlyn McGrade and Christopher Orlando, Manhattan College, Riverdale, NY***.

Pythagorean triples (a, b, c) for right triangles satisfy the relation $a^2+b^2 = c^2$, where a and b are the side lengths and c is the hypotenuse's. It's a standard result that all Pythagorean triples are of the form $a = t(m^2 - n^2)$, $b = t(2mn)$ and $c = t(m^2 + n^2)$ where $m > n$, n and t are natural numbers, m and n are coprime and not both odd and up to congruence a and b the triangle's legs can switch roles. If we consider the rational triple $(s/t, u/v, l/k)$ such that $(s/t)^2 + (u/v)^2 = (l/k)^2$ it then follows that $(svk)^2 + (utk)^2 = (ltv)^2$ which shows that all rational solutions are based on the pythagorean triples.

For an equable right triangle with sides (a, b, c), we set the perimeter P equal to its area A. That is $a + b + c = ab/2$. Now substituting the standard result for pythagorean triples stated above for a, b and c and letting t run over the positive rationals we get that $t(m^2 - n^2) + t(2mn) + t(m^2 + n^2) = t^2(mn)(m^2 - n^2)$ or $t = 2/n(m - n)$. We now have all possible equable triangles with sides $a = \frac{2(m+n)}{n}$, $b = \frac{4m}{m-n}$ and $c = \frac{2(m^2+n^2)}{n(m-n)}$ with this expression for t.

Addendum. In the special case for equable triangles with integral sides, there are exactly two non congruent triangles. Their triples are $(5, 12, 13)$ and $(6, 8, 10)$. This readily follows if we observe that for t to be a natural number then n must divide 2, which gives us that $n = 2$ and $m = 3$ or $n = 1$, and $m = 2$ and the the two triples above respectively.

Addendum. Additional remark from Dionne Bailey, Elsie Campbell, and Charles Diminnie: A related problem is to find all integer-sided triangles whose area A and perimeter P satisfy $A = mP$ for some fixed positive integer m. Some characterizations of such triangles may be found in the following sources.

- L. P. Markov, *Pythagorean Triples and the Problem $A = mP$ for Triangles*, Mathematics Magazine **79** (2006), 114–121.
- T Leong, D. Bailey, E. Campbell, C. Diminnie, and P. Swets, *Another Approach to Solving $A = mP$ for Triangles*, Mathematics Magazine **80** (2007), 363–368.

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