PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2018

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2018

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31. The randomly chosen winner for this issue is the Armstrong Problem Solvers, Armstrong Campus of Georgia Southern University. Congratulations to them and all the schools that competed (we had a record number hit the threshold this issue).

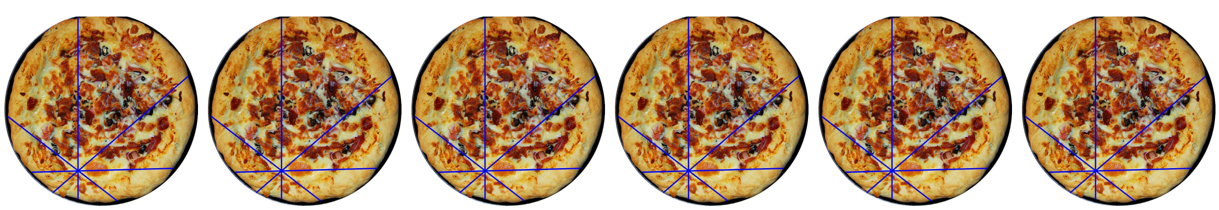

Figure 1. Pizza motivation; can you name the theorem that's represented here?

Date: March 29, 2018.

#1343: *Proposed by Ralph Morrison, Williams College.*
The following is from the 2016 Green Chicken Math Competition between Middlebury and Williams Colleges. (a) If we expand and collect like terms in $(x+y)^2$, there are three terms (since $(x+y)^2 = x^2 + 2xy + y^2$). After you expand $(w+x+y+z)^{38}$ and collect like terms, how many terms are there? (b) ***Greenie*** picks (uniformly at random) one of the terms you counted up in part (a), deletes the integer coefficient, and plugs in $w = 2$, $x = 3$, $y = 5$, and $z = 7$. What is the probability that the resulting number is divisible by 2016?

#1344: *Proposed by Steven J. Miller and Ralph Morrison, Williams College.*
The following is from the 2016 Green Chicken Math Competition between Middlebury and Williams Colleges, though it was probably inspired by a problem in the literature. In order to prove a proposition on Diophantine equations, ***Greenie*** was planning to cite Fermat's Last Theorem, which says that there are no integer solutions to $x^n + y^n = z^n$ where $n \geq 3$ and $xyz \neq 0$. However, as she is just an undergraduate chicken, she has not taken enough classes to understand the proof. She doesn't want to use a result she doesn't understand. Fortunately, she only needs a weaker version of Fermat's Last Theorem: There are no positive integer solutions x, y, z, n to $x^n + y^n = z^n$ where $n \geq z$ and $xyz \neq 0$. Please prove it, so that ***Greenie*** can complete her paper:

#1345: *Northwest Missouri State University, Problem Solving Group, Maryville, MO*
There is no general method for finding the closed-form of a convergent series. Techniques can vary from problem to problem; the following problem can be solved using methods from Calculus I and II: Find a closed form for

$$x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-3)(2n-5)\cdots 5 \cdot 3 \cdot 1}{2^n n!(2n+1)} x^{2n+1}.$$

Note: one often writes $(2n-1)!!$ for the double factorial of $2n-1$, i.e., the product of every other number from $2n-1$ down to 1.

#1346: *Proposed by Hongwei Chen, Department of Mathematics, Christopher Newport University.*
Let F_n be the n^{th} Fibonacci number with $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Define

$$a_n = 1^{F_n} 2^{F_{n-1}} \cdots (n-1)^{F_2} n^{F_1}.$$

Find a constant A such that, for sufficiently large n, $a_n \sim A^{F_{n+1}}$.

#1347: *Proposed by Zhiqi Li on behalf of Math 377 Spring 2017, Williams College.*
The following is a standard problem (though often concrete numbers are given), ending in a more open question that likely is in the literature but if so is not as well known. Let $f(n)$ be the smallest number of points on a unit circle such that no matter where those $f(n)$ points are chosen, at least n of them will be on a common closed semi-circle. Find $f(n)$. Extra credit: what would the corresponding values be on a sphere?

#1348: *Proposed by Matthew McMullen, Otterbein University.*

An equable triangle is one whose area and perimeter evaluate to the same number. Find the real number a such that there exists exactly one equable triangle with two sides of length a. (**Bonus:** Classify all pairs of real numbers (a, b) with $a \geq b$ such that there exists exactly one equable triangle with one side of length a and another side of length b.)

#1349: *Proposed by Kenny Davenport and Allen Pierce.*
In #978 (volume 11 (2000), number 3), Robert Hess asked for a 4×4 array of integers such that the four digit numbers which are the rows and the four digit numbers which are the columns are all perfect squares and the digit 0 is never used; the solution is

$$\begin{array}{cccc} 2 & 1 & 1 & 6 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 9 & 6 \\ 6 & 5 & 6 & 1 \end{array}$$

(the solution is unique). Find at least 5 solutions to the corresponding problem for a 5×5 array where again the digit 0 cannot be used. *Bonus:* if additionally the matrix is symmetric, how many solutions are there?

GRE Practice #2: *Proposed by Steven Miller, Williams College*
One of the greatest challenges students have with the math GRE subject test is that while they solve a problem, often it is faster to eliminate four wrong answers than find the exact solution (or at least eliminate a few answers, at which point on average it is advantageous to guess). Consider the following (a discussion of the answer is included after the solutions to earlier PME problems). Daneel is able to paint a house in two days, while Hari needs three days. If they work together how many days will it take? (a) 2/3 (b) 3/4 (c) 5/6 (d) 6/5 (e) 3/2.

2. Solutions

Note: After the Fall 2017 issue went to print, correct solutions were sent to several problems: #1330 by Lucas Stefanic, Rochester Institute of Technology, and #1333 by Ioannis D. Sfikas, National and Kapodistrian University of Athens.

#1335: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*
The following is from the 1993 Green Chicken Math Competition between Middlebury and Williams Colleges. At State University 7 students registered for American history, 8 students for British history, and 9 students for Chinese history. No student is allowed to take more than one history course at a time. Whenever two students from different classes get together, they decide to drop their current history courses and add the third. Otherwise there are no adds or drops. Is it possible for all students to end up in the same history course?

Solution below by ***Lucas Stefanic, Rochester Institute of Technology.*** *Also solved by the* ***Armstrong Problem Solvers, Armstrong Campus of Georgia Southern University****, and* ***Ashley Bishop, North Central College****.*

It is not possible. Let (a_1, b_1, c_1) be an ordered triple such that there are a_1 students in American history, b_1 in British history, and c_1 in Chinese history. Suppose two students from different classes get together and the triple (a_2, b_2, c_2) represents the new numbers of students in the classes.

Case 1.
Students from American and British history get together:

$$(a_2, b_2, c_2) \;=\; (a_1 - 1, b_1 - 1, c_1 + 2).$$

Students from American and Chinese history get together:

$$(a_2, b_2, c_2) \;=\; (a_1 - 1, b_1 + 2, c_1 - 1).$$

Students from British and Chinese history get together:

$$(a_2, b_2, c_2) \;=\; (a_1 + 2, b_1 - 1, c_1 - 1).$$

In every case, the congruence $a_2 - b_2 \equiv a_1 - b_1 \pmod 3$ holds. Thus if students continue to get together, then in the sequence of triples (a_n, b_n, c_n) that follow, we will always have $a_n - b_n \equiv 7 - 8 \equiv 2 \pmod 3$. But if all 24 students could end up in the same history course, then we would have $a_n - b_n \equiv 0 \pmod 3$; contradiction.

#1336: *Proposed by Pete Schumer, Middlebury College, Middlebury, VT 05753.*
The following is from the 1999 Green Chicken Math Competition between Middlebury and Williams Colleges. An integer is powerful if each of its prime factors occurs to the second power or more. Prove or disprove: There are an infinite number of pairs of consecutive powerful numbers.

Solution below by ***Zachary Morgan, Eastern Kentucky University.*** *Also solved by* ***Lucas Stefanic, Rochester Institute of Technology****.*

Using induction, we show given any pair of consecutive powerful numbers we can always find another, larger pair. For the base case, consider the powerful pair $8 = 2^3$ and $9 = 3^2$.

Now, suppose n and $n+1$ is a pair of consecutive powerful numbers and consider the consecutive numbers $4n^2 + 4n$ and $4n^2 + 4n + 1$. Note

$$4n^2 + 4n \;=\; 4(n^2 + n) \;=\; 2^2(n^2 + n) \;=\; 2^2 n(n+1)$$

is powerful if n and $n+1$ are. Similarly

$$4n^2 + 4n + 1 \;=\; (2n+1)^2$$

is powerful since every prime factor of $2n + 1$ is raised to at least the second power once it is squared.

Since n and $n+1$ are consecutive powerful numbers by the induction hypothesis, $4n^2 + 4n$ and $4n^2 + 4n + 1$ is a larger pair of consecutive powerful numbers, completing the proof.

#1337: *Proposed by Steven J. Miller, Williams College, Williamstown, MA.*
A graph G is a collection of vertices V and edges E connecting pairs of vertices. Consider the following graph. The vertices are the integers $\{2, 3, 4, \ldots, 2017\}$. Two vertices are connected by an edge if they share a divisor greater than 1; thus 30 and 1593 are connected by an edge as 3 divides each, but 30 and 49 are not. The coloring number of a graph is the smallest number of colors needed so that each vertex is colored ***and*** if two vertices are connected by an edge, then those two vertices are not colored the same. Prove the coloring number is at

least 10. What is the actual value? *This problem was first published in the Newsletter of the European Mathematical Society.*

Solution below by the ***Armstrong Problem Solvers, Armstrong Campus of Georgia Southern University.***
The coloring number of this graph is 1008. To see this, notice that exactly half of the 2016 vertices are even integers, which all share a common divisor of 2. Thus, every pair of even integers is connected by an edge, so no two even integers can have the same color, and the coloring number of the graph is at least 1008. (In the language of graph theory, the graph contains the complete graph K1008 as a subgraph.) A proper coloring with exactly 1008 different colors is easily obtained by coloring each even integer and its successor the same color. Since any two consecutive integers have a difference of 1, they must be relatively prime, and cannot share any divisors greater than one.

#1340: *Communicated by Steven J. Miller, Williams College*
Zeckendorf proved that if we define the Fibonacci numbers by $F_1 = 1, F_2 = 2$ and $F_{n+2} = F_{n+1} + F_n$ then every integer can be written uniquely as a sum of non-adjacent Fibonacci numbers. We call this the Zeckendorf decomposition; thus $2018 = 1597 + 377 + 34 + 8 + 2$. Prove this claim, and further show that if we write any N as a sum of Fibonacci numbers, no decomposition has fewer summands than the Zeckendorf decomposition.

Solution below by ***Raymond Maresca, Manhattan College.***

We first show every natural number has a Zeckendorf decomposition. First, if any natural number is a Fibonacci number, then it is its own Zeckendorf decomposition. So we take the first natural number that is not a Fibonacci number and show that it has a Zeckendorf decomposition. That is, $4 = 3 + 1$. Now we assume for all $k \leq n$, where $k, n \in \mathbb{N}$, that k has a Zeckendorf decomposition. Then the natural number $n + 1$ is either a Fibonacci number or it is not. If it is a Fibonacci number we are done, so we assume it is not. Then there exist two Fibonacci numbers F_j, F_{j+1} such that $F_j < n + 1 < F_{j+1}$ for some $1 \leq j \in \mathbb{N}$. Let the natural number $z = (n + 1) - F_j$. This allows us to conclude that $z + F_j = n + 1$ and therefore that $z < n+1$ because $F_j > 0$ for all j. Since z is a natural number less than $n+1$, the inductive hypothesis allows us to conclude that z has a Zeckendorf decomposition which we denote by $\mathsf{Zeck}(z)$. Notice $n + 1 < F_{j+1}$, allowing us to conclude that $z + F_j < F_{j+1}$. Since F_{j+1} is a Fibonacci number, we have $z + F_j < F_j + F_{j-1}$, or $z < F_{j-1}$. This allows us to conclude that F_{j-1} is not a summand of $\mathsf{Zeck}(z)$ and thus $F_j + \mathsf{Zeck}(z)$, is a Zeckendorf decomposition of $n + 1$. Therefore through induction, we have that every natural number can be written as a sum of non-adjacent Fibonacci numbers.

We now show uniqueness. We first establish the following: Consider a finite sum of non-adjacent Fibonacci numbers (i.e., a Zeckendorf decomposition) of some integer N. If F_j is the largest summand in that decomposition, then $N < F_{j+1}$, the next adjacent Fibonacci number to F_j.

Proof: We will prove this by induction on the length of the Zeckendorf decomposition. By definition of Fibonacci numbers, it is clear any Fibonacci number is less than its successor, our base case for induction. Now for all Zeckendorf decompositions, Z_1, of length $k \in \mathbb{N}$ we assume $F_k < F_{k+1}$ where F_k is the largest summand of Z_1. Notice, $Z_1 = Z_2 + F_k$ where Z_2 is a Zeckendorf decomposition consisting of the previous $k-1$ summands of Z_1. Now consider a Zeckendorf decomposition Z of length $k+1$. Then $Z = Z_1 + F_i$ where F_i is the largest summand of Z. Since the decomposition is Zeckendorf, F_k and F_i are not adjacent. Then $F_{k+1} \leq F_{i-1}$. We have

$$\begin{aligned} Z &= Z_1 + F_i \\ &= (Z_2 + F_k) + F_i \\ &< F_{k+1} + F_i && \text{(Inductive Hypothesis)} \\ &\leq F_{i-1} + F_i \\ &= F_{i+1} \end{aligned}$$

Therefore, through induction, we have for any Zeckendorf decomposition of an integer N, if F_j is the largest summand in that decomposition, then $N < F_{j+1}$, the next adjacent Fibonacci number to F_j. □

We now prove that the Zeckendorf decomposition of a natural number is unique. Suppose for any $N \in \mathbb{N}$ we have two Zeckendorf decompositions $D_1 = F_{i_1} + \cdots + F_{i_m}$ and $D_2 = F_{j_1} + \cdots + F_{j_n}$ such that $D_1 = N = D_2$. For a contradiction, suppose that the decompositions are not identical. Moreover, suppose that the summands of D_1 and D_2 are written in size order, from largest to smallest, that is $F_{j_1} > F_{j_2}$ and so on. Since the decompositions are not identical, a pair of unequal summands must occur before the shorter decomposition terminates. Say the first unequal pair occurs $k \in [1, \min\{n, m\}]$ terms into the decompositions, that is $F_{i_k} \neq F_{j_k}$ for some $k \in \mathbb{N}$. Without loss of generality, suppose $F_{i_k} < F_{j_k}$. Since D_1 is a Zeckendorf decomposition, so is $F_{i_k} + F_{i_{k+1}} + \cdots + F_{i_m}$ and from the above lemma we conclude $F_{i_k+1} > F_{i_k} + F_{i_{k+1}} + \cdots + F_{i_m}$. Since we assumed that $F_{i_k} < F_{j_k}$, we know that $F_{i_k} + F_{i_{k+1}} + \cdots + F_{i_m} < F_{i_k+1} \leq F_{j_k}$. Thus $F_{i_k} + F_{i_{k+1}} + \cdots + F_{i_m} < F_{j_k}$. Now, since the first $k-1$ pairs of summands are assumed to be equal, we have

$$\begin{aligned} D_1 &= F_{i_1} + F_{i_2} + \cdots + F_{i_k} + \cdots + F_{i_m} \\ &< F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_{j_k} && (F_{i_k} + F_{i_{k+1}} + \cdots + F_{i_m} < F_{j_k}) \\ &= F_{j_1} + F_{j_2} + \cdots + F_{j_{k-1}} + F_{j_k} && \text{(First } k-1 \text{ summands equal)} \\ &< F_{j_1} + F_{j_2} + \cdots + F_{j_k} + F_{j_{k+1}} + \cdots + F_{j_n} && \text{(Adding remaining summands of } D_2 \text{)} \\ &= D_2 \end{aligned}$$

Notice, the generality of the above result is not lost if D_1 and D_2 have different numbers of summands. Thus $D_1 < D_2$, contradicting the claim that the two decompositions are equal. Therefore the Zeckendorf decomposition of an integer is unique.

All that remains is to show that no decomposition using only Fibonacci numbers has fewer summands than the Zeckendorf one. We use the proof of the proposer, as it uses a great mathematical concept, that of a monovariant. A monovariant is a quantity that changes in at most one direction.

Given a decomposition of m into a sum of Fibonacci numbers, consider the sum of indices of terms in the decomposition (start $F_1 = 1, F_2 = 2$). If you ever have two adjacent summands you do not increase the index sum by combining (and decrease it once the smallest summand is at least F_2). If you have F_1 twice use F_2. If you have F_2 twice use F_1 and F_3. In general, if you have F_k twice use

$$2F_k \;=\; F_{k-2} + F_{k-1} + F_k \;=\; F_{k-2} + F_{k-1},$$

which has decreased the index sum for $k \geq 3$ and you now have a larger Fibonacci summand. You can only do this a bounded number of times or you'll end up with Fibonacci number larger than the largest Fibonacci number less than m, so when you terminate you cannot have any repeats or adjacencies, and thus must be a legal Zeckendorf decomposition! As you have not increased the number of summands at any step, no decomposition can have fewer summands than the Zeckendorf decomposition. (Alternatively, if you were not at a Zeckendordf decomposition you could either split a doubled index or combine two adjacent ones, lowering the index sum, and keeping the number of terms either the same or decreasing by 1.)

#1341: *Proposed by Matthew Davis, Williams College*
Consider an infinite one-dimensional board, where we may place checkers at any integer. We initialize the game by placing checkers at all the positive squares, and all their negatives; thus there are checkers only at positions $\pm 1, \pm 4, \pm 9$, and so on. As the game evolves, we may have multiple checkers at the same position (similar to how one may stack checkers in a game to make a king). At each step, you can perform one of several moves.

- You can either add or remove any finite number of checkers.
- Given integers a, k with $k > 1$, you may add a checker at each position ak^n (where n ranges over the non-negative integers).
- If there are integers a, k with $k > 1$ such that for all n there is always at least one checker at position ak^n, then you may remove one checker from each of these positions.

Is it possible to move every checker inward one space in a finite number of moves? In other words, can we reach the state where there are two checkers at 0, and then one each at $\pm 3, \pm 8, \pm 15, \dots$.

Solution below by ***Lucas Stefanic, Rochester Institute of Technology.***
It is not possible. To do so would require removing all checkers from the positions p^2, where p ranges over all primes. To prove that this task is impossible in a finite number of moves, it suffices to show that at every step, we can only remove checkers from finitely many of these positions.

Obviously, the first two move choices cannot remove checkers from infinitely many p^2 positions, so we focus on the third choice. Let a, k be integers with $a > 0, k > 1$, and consider the integers $a, ak, ak^2, \dots$. Suppose that these integers include two squares of primes p_1^2, p_2^2 with $p_1^2 < p_2^2$. Then $(p_2/p_1)^2$ is some power of k, and therefore an integer. But since p_1

and p_2 are distinct primes, this is impossible. Hence, the third move choice can only remove checkers from at most one p^2 position.

#1342: *Proposed by Ralph Morrison, Williams College*
The following is from the 2016 Green Chicken Math Competition between Middlebury and Williams Colleges. While attending a concert, ***Greenie*** was instructed by the performer to "call her, maybe." Unfortunately ***Greenie*** can't remember the performer's Skype name exactly, but she remembers noticing that it had no repeated digits, was divisible by 3 but not by 6 or by 9, and that it was the largest possible integer satisfying all those properties. What was the number?

Solution below by ***Lucas Stefanic, Rochester Institute of Technology.*** *Also solved by the* ***Armstrong Problem Solvers, Armstrong Campus of Georgia Southern University****,* ***Matthew McCollum, North Central College****, the* ***Skidmore College Problem Group****, and* ***Zachary Morgan, Eastern Kentucky University****.*
The number is 987654201.

Recall a positive integer is congruent modulo 9 to the sum of its digits. It's easily verified that 987654201 has no repeated digits and is divisible by 3, but not by 6 or 9. Suppose to the contrary that N is a larger positive integer which satisfies these properties.

If N has all 10 digits from 0 to 9, then $N \equiv 45 \equiv 0 \bmod 9$. Thus N must have only 9 digits (we know it has at least 9 digits, as 987654201 works).

For N to be divisible by 3 but not 9, we need the sum of its digits to be divisible by 3 but not 9. The one digit N does not have must then be 3 or 6. If N is missing 6, then N is at most 987543210. Thus N is missing 3.

We have that N is at most 987654210, and we are assuming it is larger than 987654201. The only candidates left are 987654204, 987654207, and 987654210. None of these integers satisfy all the desired properties (the first two do not have distinct digits, while the third is a multiple of 6); our claim follows by contradiction.

GRE Practice #2: *Proposed by Steven Miller, Williams College*

One of the greatest challenges students have with the math GRE subject test is that while they solve a problem, often it is faster to eliminate four wrong answers than find the exact solution (or at least eliminate a few answers, at which point on average it is advantageous to guess). Consider the following (a discussion of the answer is included after the solutions to earlier PME problems). Daneel is able to paint a house in two days, while Hari needs three days. If they work together how many days will it take? (a) 2/3 (b) 3/4 (c) 5/6 (d) 6/5 (e) 3/2.

Solution: We can solve directly. Daneel paints 1/2 of a house in a day, while Hari paints 1/3. Thus if d is the number of days needed to paint a house, we have $d \cdot 1/2 + d \cdot 1/3 = 1$, or $d \cdot 5/6 = 1$ and thus $d = 6/5$.

While this wasn't too bad to solve, I've elected to discuss this problem as it illustrates a great principle. Specifically, we can easily come up with a tight enough range on where the answer lives so that we do not need to solve it exactly. As Daneel paints faster than Hari, the answer must be strictly between how long it would take if there were two Daneels (which is 1 day) and two Haris (which is 3/2 days). Notice only one of the five answers is in this region; thus there is no need to find the true answer, as we can eliminate four of the five options.

E-mail address: sjm1@williams.edu

Associate Professor of Mathematics, Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267