# PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2018 

STEVEN J. MILLER (EDITOR)

## 1. Problems: Fall 2018

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk $\left({ }^{*}\right)$ preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $\$ 100$ ). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31. Congratulations to the Missouri State University Problem Solving Group and Columbus State University, which both qualified; the randomly selected winner was the Missouri State University Problem Solving Group.


Figure 1. Pizza motivation; can you name the theorem that's represented here?
\#1350: Proposed by Blake Mackall.
The following problem is inspired by a sports modeling problem. A function $f$ is a continuous probability density if it is non-negative and integrates to 1 (and, of course, is continuous). Given a continuous function $\phi:[0,1] \rightarrow(0, \infty)$, find a continuous probability density $f$ : $[0,1] \rightarrow[0, \infty)$ which minimizes $\int_{0}^{1} \phi(x) f^{2}(x) d x$.
\#1351: Proposed by Hongwei Chen, Christopher Newport University.
Let $\zeta$ be the Riemann zeta function; $\zeta(3)=\sum_{n=1}^{\infty} 1 / n^{3}=1.2020569 \ldots$ is now known as Apéry constant since Roger Apéry first proved that $\zeta(3)$ is irrational in 1979. Since then, more efforts have been focused on seeking a rapidly convergent series. Show that

$$
\zeta(3)=1+\sum_{n=1}^{\infty} \frac{1}{n^{3}+4 n^{7}}
$$

This series has the convergence rate $O\left(n^{-7}\right)$ instead of $O\left(n^{-3}\right)$.
\#1352: Proposed by Pete Schumer, Middlebury College.
The following is from the 2017 Green Chicken Math Competition between Middlebury and Williams Colleges. If the length of the side of a triangle is less than the average lengths of the other two sides, show that the opposite angle is less than the average of the other two angles.

## \#1353: Proposed by Kenny Davenport.

Let $C_{n}=\frac{1}{n+1}\binom{2 n}{n}$ be the $n^{\text {th }}$ Catalan number, and $L_{n}$ equal the $n^{\text {th }}$ Lucas number (these are given by the recurrence $L_{n+1}=L_{n}+L_{n-1}$ with initial conditions $L_{0}=2$ and $L_{1}=1$ ). Find

$$
\sum_{n=0}^{\infty} n C_{n+1} L_{n} / 8^{n} .
$$

\#1354: David Benko, University of South Alabama.
There is a road around a lake with 9 gas stations along it, whose locations can be arbitrarily fixed. We have a map of the road which also states how much gasoline is at each gas station. We know how many miles per gallon our car gets, and interestingly it turns out that the total amount of gasoline at the stations is exactly enough to go around the lake once. Starting with an empty tank, can we always choose a gas station such that if a helicopter (carefully!) drops our car at that gas station, we can use it as a starting point to go around the lake, arriving back to the initial gas station? We assume that there are no other cars on the road, the gasoline is free, the size of our gasoline tank is unlimited, and we always get the same miles per gallon throughout our trip.
\#1355: David Benko, atthew McMullen, Otterbein University, Westerville, OH. Suppose $\lim _{n \rightarrow \infty} a_{n}=0$. Does it necessarily follow that $\sum \frac{1}{n^{1+a_{n}}}$ diverges? Prove or give a counterexample.

## GRE Practice \#3: From ETS GRE Mathematics Practice Book

One of the greatest challenges students have with the math GRE subject test is that while they solve a problem, often it is faster to eliminate four wrong answers than find the exact solution (or at least eliminate a few answers, at which point on average it is advantageous to guess). Consider the following (a discussion of the answer is included after the solutions to earlier PME problems), taken from one of the on-line collections of GRE problems (it was Problem 3). Find

$$
\int_{e^{-3}}^{e^{-2}} \frac{d x}{x \log x}
$$

The choices are (a) 1 , (b) $2 / 3$, (c) $3 / 2$, (d) $\log (2 / 3), \quad$ (e) $\log (3 / 2)$.
Special Bonus: The Four Four Game: The Four Fours game (or problem) asks you to represent as many numbers as you can using exactly four fours (though some variants just require you to use at most four fours). For example,

$$
1=\frac{44}{44} \quad \text { or } \quad(4 / 4)^{4 * 4} \quad \text { or } \quad \sqrt[4]{4^{4}} / 4
$$

to name a few. A colleague of mine, Steve Conrad of http://www.mathleague.com/, has collected some of his favorites; see the Spring 2019 issue for his collection.

## 2. Solutions

We begin with a comment by Benjamin Dickman, Ph.D., Math Teacher and Math Coach at The Hewitt School in New York, NY, on Problem 1254(d): For $n>2$, are $9^{n}-2$ and $9^{n}+2$ ever both prime?. The published solution, which appeared in the Fall 2012 issue, contains both typographical and mathematical errors, as pointed out in the comments of the following MathOverflow answer: https://mathoverflow.net/q/105004.

For example, the claim that $p=1+(20 a+1) 6$ implies 7 divides $n$ has a typographical error (it should say 7 divides $p$ ) and a mathematical error: If $p$ is of this form, then there is no reason to expect that 7 divides it. For example, when $a=1$, we have $p=1+(121) 6=727$, which is not divisible by 7 . Part (d) of this problem appears to be open, and it was submitted without solution by the proposer. Parts (a) (b) and (c) are comparatively straightforward, with a solution to part (c) subsuming parts (a) and (b).
\#1343: Proposed by Ralph Morrison, Williams College.
The following is from the 2016 Green Chicken Math Competition between Middlebury and Williams Colleges. (a) If we expand and collect like terms in $(x+y)^{2}$, there are three terms (since $\left.(x+y)^{2}=x^{2}+2 x y+y^{2}\right)$. After you expand $(w+x+y+z)^{38}$ and collect like terms, how many terms are there? (b) Greenie picks (uniformly at random) one of the terms you counted up in part (a), deletes the integer coefficient, and plugs in $w=2, x=3, y=5$, and $z=7$. What is the probability that the resulting number is divisible by 2016 ?

Solution below by Levent Batakci, Hershey High School, PA. Also solved by the Missouri State University Problem Solving Group and Zach Folta, Columbus State University.

We solve (a) first. If we ignore coefficients, each distinct term in the expansion of (w+ $x+y+z)^{38}$ has the form

$$
w^{a_{1}} x^{a_{2}} y^{a_{3}} z^{a_{4}}
$$

where $a_{1}, a_{2}, a_{3}, a_{4} \in \mathbb{N} \cup\{0\}$.
We want to find the number of solutions to $a_{1}+a_{2}+a_{3}+a_{4}=38$ where $a_{1}, a_{2}, a_{3}, a_{4} \in$ $\mathbb{N} \cup\{0\}$. This is equivalent to counting how many ways we can divide 38 indistinguishable cookies among 4 distinguishable people. One way of enumerating this is creating 41 positions on a line to place 38 cookies and 3 dividers (number of the people minus 1). The positions of the dividers will determine the number of cookies each person gets. These 3 dividers can be placed in the 41 positions in

$$
\binom{38+3}{3}=\frac{41!}{3!38!}=10660 \text { ways. }
$$

Hence, there are 10660 terms in $(w+x+y+z)^{38}$.
For (b), note the prime factorization of 2016 is $2^{5} \cdot 3^{2} \cdot 7$. Given that $(w, x, y, z)=(2,3,5,7)$, each term of the expansion (with its integer coefficient deleted) has the form $2^{a_{1}} 3^{a_{2}} 5^{a_{3}} 7^{a_{4}}$. This implies we want to find all possible solutions to $a_{1}+a_{2}+a_{3}+a_{4}=38$ where $a_{1} \geq$ $5, a_{2} \geq 2, a_{3} \geq 0, a_{4} \geq 1$. Our constraints imply that the number of ways is equivalent to the number of ways to distribute 30 cookies among 4 people; the reduction is because $5+2$ $+1=8$ of the cookies have already been assigned. By the same argument used in part (a), there are

$$
\binom{30+3}{3}=\frac{33!}{3!30!}=5456
$$

coefficient-less terms that are divisible by 2016. Since there are 10660 terms in total, the probability of picking one that is divisible by 2016 uniformly at random is

$$
\frac{5456}{10660}=\frac{1364}{2665} \approx 0.51182
$$

\#1344: Proposed by Steven J. Miller and Ralph Morrison, Williams College.
The following is from the 2016 Green Chicken Math Competition between Middlebury and Williams Colleges, though it was probably inspired by a problem in the literature. In order to prove a proposition on Diophantine equations, Greenie was planning to cite Fermat's Last Theorem, which says that there are no integer solutions to $x^{n}+y^{n}=z^{n}$ where $n \geq 3$ and $x y z \neq 0$. However, as she is just an undergraduate chicken, she has not taken enough classes to understand the proof. She doesn't want to use a result she doesn't understand. Fortunately, she only needs a weaker version of Fermat's Last Theorem: There are no positive integer solutions $x, y, z, n$ to $x^{n}+y^{n}=z^{n}$ where $n \geq z$ and $x y z \neq 0$. Please prove it, so that Greenie can complete her paper!

Solution below by Panagiotis T. Krasopoulos, Unified Social Security Fund, Athens, Greece. Also solved by Missouri State University Problem Solving Group (who proved more; their stronger result is also included).

Since $z \leq n$ we have that $z \in\{1,2, \ldots, n\}$ and of course $x, y \in\{1,2, \ldots, z-1\}$ (because if either $x$ or $y$ is $\geq z$, then $x^{n}+y^{n}>z^{n}$ ). In order to see that there are no integer solutions
of the equation, we will prove that

$$
x^{n}+y^{n}<z^{n}
$$

for the above values of $x, y, z$ and $n \geq 3$ a given integer. It is clear that the maximum value of $x^{n}+y^{n}$ which is also a potential solution of the equation is equal to $2(z-1)^{n}$. So it is enough to prove

$$
2(z-1)^{n}<z^{n}
$$

equivalently

$$
\left(1-\frac{1}{z}\right)^{n}<\frac{1}{2}
$$

It is also true that for $z \in\{1,2, \ldots, n\}$ it holds $\left(1-\frac{1}{z}\right)^{n} \leq\left(1-\frac{1}{n}\right)^{n}$, so we have to prove that

$$
\left(1-\frac{1}{n}\right)^{n}<\frac{1}{2}
$$

Instead of proving the above inequality, we will prove that $\left(1-\frac{1}{n}\right)^{n} \leq 1 / e$ for $n \geq 3$. Equivalently we will prove that

$$
\ln \left(1-\frac{1}{n}\right)^{n} \leq \ln \frac{1}{e}
$$

or

$$
\ln \left(1-\frac{1}{n}\right)+\frac{1}{n} \leq 0
$$

Now, since $n \geq 3$ we have $0<1 / n \leq 1 / 3$ and by setting $x=1 / n$ we can define the function $f(x)=\ln (1-x)+x$ for $x \in[0,1 / 3]$. So, we want to prove that $f(x) \leq 0$ for $x \in[0,1 / 3]$. First we observe that $f(0)=0$. Secondly, we will show that $f$ is decreasing in $[0,1 / 3]$. We have

$$
f^{\prime}(x)=\frac{x}{x-1} \leq 0
$$

for $x \in[0,1 / 3]$. Thus, $f(x) \leq 0$ for $x \in[0,1 / 3]$. So, the inequality $\left(1-\frac{1}{n}\right)^{n} \leq 1 / e<1 / 2$ holds for $n \geq 3$. The proof is complete.

We claim, slightly more generally, that there are no such solutions with $z<n / \ln 2 \approx$ $1.4427 n$

We show that $2^{1 / n} /\left(2^{1 / n}-1\right)>n / \ln 2$, for $n \geq 1$. We then claim that for $1 \leq z<n / \ln 2$, $f(z)=z^{n}-2(z-1)^{n}>0$. This follows from the fact that the only roots of $z^{n}-2(z-1)^{n}=0$ are $z=2^{1 / n} /\left(2^{1 / n}-1\right)$ when $n$ is odd and $z=2^{1 / n} /\left(2^{1 / n}-1\right)$ or $z=2^{1 / n} /\left(2^{1 / n}+1\right)$ when $n$ is even. Now $2^{1 / n} /\left(2^{1 / n}+1\right)<1$ and $2^{1 / n} /\left(2^{1 / n}-1\right)>n / \ln 2$ by the first claim. By the Intermediate Value Theorem, $f(z)$ has the same sign throughout $[1, n / \ln 2)$ and $f(1)=1$, so $f(z)>0$ on the interval. Finally, since $x y z \neq 0, z \geq 1$ and $x, y \leq z-1$. But then $z^{n}=x^{n}+y^{n} \leq 2(z-1)^{n}$, and this contradicts the preceding inequality.

From the Missouri State University Problem Solving Group: We claim, slightly more generally, that there are no such solutions with $z<n / \ln 2 \approx 1.4427 n$. First show that $2^{1 / n} /\left(2^{1 / n}-1\right)>n / \ln 2$, for $n \geq 1$. Then for $1 \leq z<n / \ln 2$, note $f(z)=z^{n}-2(z-1)^{n}>0$; this follows from the fact that the only roots of $z^{n}-2(z-1)^{n}=0$ are $z=2^{1 / n} /\left(2^{1 / n}-1\right)$ when
$n$ is odd and $z=2^{1 / n} /\left(2^{1 / n}-1\right)$ or $z=2^{1 / n} /\left(2^{1 / n}+1\right)$ when $n$ is even. Now $2^{1 / n} /\left(2^{1 / n}+1\right)<1$ and $2^{1 / n} /\left(2^{1 / n}-1\right)>n / \ln 2$ by the first claim. By the Intermediate Value Theorem, $f(z)$ has the same sign throughout $[1, n / \ln 2)$ and $f(1)=1$, so $f(z)>0$ on the interval. Finally, since $x y z \neq 0, z \geq 1$ and $x, y \leq z-1$. But then $z^{n}=x^{n}+y^{n} \leq 2(z-1)^{n}$, contradicting the preceding inequality.
\#1345: Northwest Missouri State University, Problem Solving Group, Maryville, MO There is no general method for finding the closed-form of a convergent series. Techniques can vary from problem to problem; the following problem can be solved using methods from Calculus I and II: Find a closed form for

$$
x+\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2 n-3)(2 n-5) \cdots 5 \cdot 3 \cdot 1}{2^{n} n!(2 n+1)} x^{2 n+1}
$$

Note: one often writes $(2 n-1)$ !! for the double factorial of $2 n-1$, i.e., the product of every other number from $2 n-1$ down to 1 .

Solution below by Missouri State University Problem Solving Group. Also solved by Volkhard Schindler and Kenneth Davenport.

We claim that the sum is

$$
\frac{1}{2}\left(x \sqrt{1+x^{2}}+\operatorname{arcsinh}(x)\right)
$$

within the interval of convergence (which is $[-1,1]$ ).
By the generalized binomial theorem

$$
\begin{aligned}
x \sqrt{1+x^{2}} & =x\left(1+x^{2}\right)^{1 / 2} \\
& =x\left(1+\sum_{n=1}^{\infty}\binom{1 / 2}{n} x^{2 n}\right) \\
& =x+\sum_{n=1}^{\infty} \frac{(1 / 2)(-1 / 2)(-3 / 2) \ldots(-(2 n-3) / 2)}{n!} x^{2 n+1} \\
& =x+\sum_{n=1}^{\infty}(-1)^{n-1} \frac{(2 n-3)!!}{2^{n} n!} x^{2 n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\sqrt{1+t^{2}}} & =\left(1+t^{2}\right)^{-1 / 2} \\
& =1+\sum_{n=1}^{\infty}\binom{-1 / 2}{n} t^{2 n} \\
& =1+\sum_{n=1}^{\infty} \frac{(-1 / 2)(-3 / 2)(-5 / 2) \ldots(-(2 n-1) / 2)}{n!} t^{2 n} \\
& =1+\sum_{n=1}^{\infty}(-1)^{n} \frac{(2 n-1)!!}{2^{n} n!} t^{2 n} .
\end{aligned}
$$

Therefore (the order of the sum and the integral can be interchanged as the integral of the sum of the absolute values is finite)

$$
\begin{aligned}
\operatorname{arcsinh}(x) & =\int_{0}^{x} \frac{1}{\sqrt{1+t^{2}}} d t \\
& =\int_{0}^{x} 1+\sum_{n=1}^{\infty}(-1)^{n} \frac{(2 n-1)!!}{2^{n} n!} t^{2 n} d t \\
& =x+\sum_{n=1}^{\infty}(-1)^{n} \frac{(2 n-1)!!}{2^{n} n!(2 n+1)} x^{2 n+1}
\end{aligned}
$$

Finally

$$
\begin{aligned}
\frac{1}{2}\left(x \sqrt{1+x^{2}}+\operatorname{arcsinh}(x)\right)= & \frac{1}{2}\left(x+\sum_{n=1}^{\infty}(-1)^{n-1} \frac{(2 n-3)!!}{2^{n} n!} x^{2 n+1}\right. \\
& \left.+x+\sum_{n=1}^{\infty}(-1)^{n} \frac{(2 n-1)!!}{2^{n} n!(2 n+1)} x^{2 n+1}\right) \\
= & x+\frac{1}{2} \sum_{n=1}^{\infty}(-1)^{n-1} \frac{(2 n-3)!!}{2^{n} n!}\left(1-\frac{(2 n-1)}{2 n+1}\right) x^{2 n+1} \\
= & x+\frac{1}{2} \sum_{n=1}^{\infty}(-1)^{n-1} \frac{(2 n-3)!!}{2^{n} n!} \cdot \frac{2}{2 n+1} x^{2 n+1} \\
= & x+\sum_{n=1}^{\infty}(-1)^{n-1} \frac{(2 n-3)!!}{2^{n} n!(2 n+1)} x^{2 n+1}
\end{aligned}
$$

as claimed.
\#1346: Proposed by Hongwei Chen, Department of Mathematics, Christopher Newport University.
Let $F_{n}$ be the $n^{\text {th }}$ Fibonacci number with $F_{1}=F_{2}=1$, and $F_{n}=F_{n-1}+F_{n-2}$ for $n>2$. Define

$$
\begin{equation*}
a_{n}=\frac{A^{F_{n+1}}}{n} b_{n} . \tag{2.1}
\end{equation*}
$$

Find a constant $A$ such that, for sufficiently large $n, a_{n} \sim A^{F_{n+1}}$.
Solution below by E. Ionascu and Christopher Lanen, Columbus State University.
Solution: We will show that the constant $A$ for which the sequence $\left\{b_{n}\right\}$ defined by (2.1) is convergent to 1 is uniquely determined by this condition and it is given by

$$
\begin{equation*}
A=\exp \left(\sum_{k=0}^{\infty} \frac{1}{\phi^{k}} \ln \left(1+\frac{1}{k+1}\right)\right) \approx 3.200960647 \tag{2.2}
\end{equation*}
$$

where $\phi=\frac{1+\sqrt{5}}{2}>1$ is the Golden ratio. Clearly the series in 2.2 is convergent by the comparison test, with respect to the convergent geometric progression series $\sum_{k=0}^{\infty} \frac{1}{\phi^{k}}=\frac{1}{1-\frac{1}{\phi}}$.

From (2.1), we see that $\ln a_{n}-F_{n+1} \ln A+\ln n=\ln b_{n}$, and so $b_{n} \rightarrow 1$ if and only if

$$
\begin{equation*}
\ln a_{n}-F_{n+1} \ln A+\ln n \rightarrow 0 . \tag{2.3}
\end{equation*}
$$

By Binet's formula $F_{n}=\frac{\phi^{n}-\left(\frac{-1}{\phi}\right)^{n}}{\sqrt{5}}, F_{n+1}$ converges to infinity and

$$
\lim _{n \rightarrow \infty} \frac{\ln n}{F_{n+1}}=\lim _{n \rightarrow \infty} \sqrt{5} \frac{\ln n}{\phi^{n+1}\left[1-\left(\frac{-1}{\phi^{2}}\right)^{n+1}\right]}=\lim _{n \rightarrow \infty} \sqrt{5} \frac{\ln n}{\phi^{n+1}}=\lim _{x \rightarrow \infty} \sqrt{5} \frac{\ln x}{\phi^{x+1}}=0
$$

The last limit can be shown to be true, using for instance L'Hospital's Rule. From (2.3), dividing by $F_{n+1}$, we get that

$$
\ln A=\lim _{n \rightarrow \infty} \frac{\ln a_{n}}{F_{n+1}}=\lim _{n \rightarrow \infty} \frac{\sum_{k=1}^{n} F_{k} \ln (n+1-k)}{F_{n+1}}
$$

In this limit, because the denominator is strictly increasing and convergent to infinity, we can apply Stolz-Cesaro's Lemma yielding:

$$
\ln A=\lim _{n \rightarrow \infty} \frac{\sum_{k=1}^{n+1} F_{k} \ln (n+2-k)-\sum_{k=1}^{n} F_{k} \ln (n+1-k)}{F_{n+2}-F_{n+1}} \Leftrightarrow
$$

$\ln A=\lim _{n \rightarrow \infty} \frac{F_{n+1} \ln (1)+\sum_{k=1}^{n} F_{k} \ln (n+2-k)-\sum_{k=1}^{n} F_{k} \ln (n+1-k)}{F_{n}}=\lim _{n \rightarrow \infty} \frac{\sum_{k=1}^{n} F_{k} \ln \frac{n+2-k}{n+1-k}}{F_{n}}$.
At this point, we can use Binet's formula and get

$$
\begin{aligned}
\ln A & =\lim _{n \rightarrow \infty} \frac{\sum_{k=1}^{n}\left(\phi^{k}-\left(\frac{-1}{\phi}\right)^{k}\right) \ln \frac{n+2-k}{n+1-k}}{\phi^{n}-\left(\frac{-1}{\phi}\right)^{n}}=\lim _{n \rightarrow \infty} \frac{\sum_{k=1}^{n}\left(\phi^{k-n}-\frac{(-1)^{k}}{\phi^{k+n}}\right) \ln \left(1+\frac{1}{n+1-k}\right)}{1-\left(\frac{-1}{\phi^{2}}\right)^{n}} \Leftrightarrow \\
\ln A & =\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \frac{1}{\phi^{n-k}} \ln \left(1+\frac{1}{n+1-k}\right)-\lim _{n \rightarrow \infty} \sum_{k=1}^{n} \frac{(-1)^{k}}{\phi^{n+k}} \ln \left(1+\frac{1}{n+1-k}\right) .
\end{aligned}
$$

In the first limit above, we change the index of summation $n-k=j$ and in the second we observe that

$$
\left|\sum_{k=1}^{n} \frac{(-1)^{k}}{\phi^{n+k}} \ln \left(1+\frac{1}{n+1-k}\right)\right| \leq \sum_{k=1}^{n} \frac{\left|(-1)^{k}\right|}{\phi^{n+k}} \frac{1}{n+1-k}
$$

since $\ln (1+x) \leq x$ for all $x \geq 0$. So, we conclude that

$$
\left|\sum_{k=1}^{n} \frac{(-1)^{k}}{\phi^{n+k}} \ln \left(1+\frac{1}{n+1-k}\right)\right| \leq \sum_{k=1}^{n} \frac{1}{\phi^{n+k}}<\frac{1}{\phi^{n+1}} \sum_{s=0}^{\infty} \frac{1}{\phi^{s}}=\frac{1}{\phi^{n+1}} \frac{1}{1-\frac{1}{\phi}} \rightarrow 0
$$

Then, we obtain

$$
\ln A=\lim _{n \rightarrow \infty} \sum_{j=0}^{n-1} \frac{1}{\phi^{j}} \ln \left(1+\frac{1}{j+1}\right)=\sum_{k=0}^{\infty} \frac{1}{\phi^{k}} \ln \left(1+\frac{1}{k+1}\right)
$$

and so the claim in $\sqrt{2.2}$ follows.

In order to finish this proof, we still have to prove (2.3) for this choice of $A$. Let us observe that

$$
\begin{align*}
\ln \left(a_{n}\right) & =\sum_{k=1}^{n} F_{k} \ln (n+1-k)=\sum_{k=1}^{n}\left(F_{k+2}-F_{k+1}\right) \ln (n+1-k) \\
& =\sum_{k=1}^{n} F_{k+2} \ln (n+1-k)-\sum_{k=1}^{n} F_{k+1} \ln (n+1-k) . \tag{2.4}
\end{align*}
$$

Changing the index of summation in the sum $\sum_{k=1}^{n} F_{k+2} \ln (n+1-k)$ gives

$$
\sum_{k=2}^{n+1} F_{k+1} \ln (n+2-k)=F_{n+2} \ln (1)-\ln (n+1)+\sum_{k=1}^{n} F_{k+1} \ln (n+2-k)
$$

Then, we can continue

$$
\ln \left(a_{n}\right)+\ln n=-\ln \left(1+\frac{1}{n}\right)+\sum_{k=1}^{n} F_{k+1} \ln \left(1+\frac{1}{n+1-k}\right) .
$$

Using Binet's formula this leads to
$\ln \left(a_{n}\right)+\ln n=-\ln \left(1+\frac{1}{n}\right)+\frac{1}{\sqrt{5}} \sum_{k=1}^{n} \phi^{k+1} \ln \left(1+\frac{1}{n+1-k}\right)-\frac{1}{\sqrt{5}} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{\phi^{k+1}} \ln \left(1+\frac{1}{n+1-k}\right)$.
The first sum above can be written in the following way:
$\sum_{k=1}^{n} \phi^{k+1} \ln \left(1+\frac{1}{n+1-k}\right)=\phi^{n+1} \sum_{k=1}^{n} \phi^{k-n} \ln \left(1+\frac{1}{n+1-k}\right)=\phi^{n+1} \sum_{j=0}^{n-1} \frac{1}{\phi^{j}} \ln \left(1+\frac{1}{j+1}\right)$.
The second sum can be transformed in a similar way, but with an upper bound:

$$
\left|\sum_{k=1}^{n} \frac{(-1)^{k+1}}{\phi^{k+1}} \ln \left(1+\frac{1}{n+1-k}\right)\right| \leq \frac{1}{\phi^{n+1}} \sum_{j=0}^{n-1} \phi^{j} \ln \left(1+\frac{1}{j+1}\right) \rightarrow 0
$$

the last limit being a consequence of Stolz-Cesaro's Lemma again. So, in order to prove (2.3), we need to show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\ln \left(a_{n}\right)-F_{n+1} \ln A+\ln n\right)=\lim _{n \rightarrow \infty} \frac{(-1)^{n+1}}{\sqrt{5}} \frac{1}{\phi^{n+1}} \ln A-\lim _{n \rightarrow \infty} \frac{1}{\sqrt{5}} \phi^{n+1} \sum_{j=n}^{\infty} \frac{1}{\phi^{j}} \ln \left(1+\frac{1}{j+1}\right) \tag{2.5}
\end{equation*}
$$

The first limit is clearly equal to zero, and for the second limit let us observe that

$$
\phi^{n+1} \sum_{j=n}^{\infty} \frac{1}{\phi^{j}} \ln \left(1+\frac{1}{j+1}\right) \leq \frac{1}{\phi(n+1)}\left(1+\frac{1}{\phi}+\frac{1}{\phi^{2}}+\cdots\right)=\frac{1}{\phi(n+1)} \frac{1}{1-\frac{1}{\phi}} \rightarrow 0
$$

and therefore, (2.3) is correct.
Remark: We have the following improved asymptotic formula

$$
\begin{equation*}
a_{n}=\frac{A^{F_{n+1}}}{n}\left(1+\frac{L}{n}+o(1 / n)\right), L=\text { something in terms of } \sqrt{5} \text { and } \phi . \tag{2.6}
\end{equation*}
$$

\#1347: Proposed by Zhiqi Li on behalf of Math 377 Spring 2017, Williams College.
The following is a standard problem (though often concrete numbers are given), ending in a more open question that likely is in the literature but if so is not as well known. Let $f(n)$ be the smallest number of points on a unit circle such that no matter where those $f(n)$ points are chosen, at least $n$ of them will be on a common closed semi-circle. Find $f(n)$. Extra credit: what would the corresponding values be on a sphere?

Solution below by Mohammad Ozaslan, University of Colorado Boulder. Also solved by Elizabeth Waye, College at Brockport, SUNY.
The answer is

$$
f(n)= \begin{cases}1, & n=1 \\ 2 n-2, & n \geq 2\end{cases}
$$

Proof. If $n=1$, then 1 point on the circle clearly guarantees that one point lies on a closed semicircle. We look at the case when $n \geq 2$.

Let $O$ be the center of the unit circle, and suppose we are given $2 n-2$ distinct points on the circle. Label a point $A$, and draw the line $O A$ to divide the circle into two closed semicircles. We will denote them by $\alpha, \beta$. Note that $A$ is on both of these closed semicircles, and that we have $2 n-3$ points on the circle, excluding $A$.

Case 1. If $\alpha$ has at least $n-1$ of these points, by including $A$ we will have at least $n$ points on $\alpha$.

Case 2. Otherwise, $\alpha$ has at most $n-2$ of these points. Thus, $\beta$ has at least $(2 n-3)-(n-2)=$ $n-1$ of these points. By including $A, \beta$ has at least $n$ points.

In any case, there exists a closed semicircle with at least $n$ points. Therefore,

$$
f(n) \leq 2 n-2
$$

Now, we will show that $2 n-3$ points can be chosen on a unit circle such that $n$ points do not appear on any closed semicircle. If $n=2$, then $2 n-3=1$, so 2 points do not even appear on the circle, and thus do not appear on a semicircle. Otherwise, construct a regular $(2 n-3)$-gon, with the points incident to our unit circle. Note that the angle between adjacent points is $\frac{2 \pi}{2 n-3}$.

Assume, for contradiction, that at least $n$ points appear on some closed semicircle. Then, the sum of the angles between adjacent points on the semicircle should be less than or equal to $\pi$. Now, choose $n$ adjacent points in this closed semicircle, such that we get a path if we look only at these points and the edges between them on the $(2 n-3)$-gon. With these points, note that exactly $n-1$ angles between adjacent points exist. Then, the sum of these
angles is $(n-1) \frac{2 \pi}{2 n-3}$, and we must have

$$
\begin{aligned}
(n-1) \frac{2 \pi}{2 n-3} & \leq \pi \\
\frac{2 n-2}{2 n-3} & \leq 1 \\
2 n-2 & \leq 2 n-3 \\
3 & \leq 2, \text { a contradiction. }
\end{aligned}
$$

Hence, $n$ points cannot appear on some closed semicircle, so $f(n) \neq 2 n-3$.
We want to use this fact to show that $f(n)>2 n-3$. Assume, for contradiction, that $f(n)=m$, where $m \leq 2 n-3$. Then, construct $2 n-3$ points on a unit circle as we did previously, and choose any $m$ points on the circle. Since $f(n)=m, n$ of these $m$ points must appear on some closed semicircle, by definition. However, we have just proved that this is not possible, and thus have a contradiction.

Thus, $f(n)>2 n-3$. Since $f(n) \leq 2 n-2$, we have that $f(n)=2 n-2$, for $n \geq 2$. Therefore,

$$
f(n)= \begin{cases}1, & n=1 \\ 2 n-2, & n \geq 2\end{cases}
$$

\#1349: Proposed by Kenny Davenport and Allen Pierce.
In \#978 (volume 11 (2000), number 3), Robert Hess asked for a $4 \times 4$ array of integers such that the four digit numbers which are the rows and the four digit numbers which are the columns are all perfect squares and the digit 0 is never used; the solution is

| 2 | 1 | 1 | 6 |
| :--- | :--- | :--- | :--- |
| 1 | 2 | 2 | 5 |
| 1 | 2 | 9 | 6 |
| 6 | 5 | 6 | 1 |

(the solution is unique). Find at least 5 solutions to the corresponding problem for a $5 \times 5$ array where again the digit 0 cannot be used. Bonus: if additionally the matrix is symmetric, how many solutions are there?

Solution below by Haile Gilroy, McNeese State University, Lake Charles, LA. Also solved by Jason L. Brown, CSU - Columbus, Georgia.
First, create a list of the squares of integers from 100 to 316 . These are the perfect squares which are five digits long. Then, delete all perfect squares on the list containing zeros.

Notice that a perfect square must end in one of the following digits: $1,4,5,6$, or 9 , not including zero. So, the fifth row and column of the desired array may only contain these digits. There are only fifteen numbers on the list meeting this criteria, so these are the only possiblities for the fifth row or column of a solution.

Next, notice from the list of perfect squares that a number with 1,4 , or 9 in the ones place must have an even number in the tens place. A number with 5 in the ones place must have a 2 in the tens place, and a number with 6 in the ones place must have an odd number in the
tens place. This creates patterns of even and odd digits in the fourth row and column, which can be used to eliminate multiple fifth row-column pairings. For example, if 4 and 6 are the fourth digits of the fifth row and column, respectively, this pairing may be eliminated, as the fourth digit of the fourth row or column cannot be both even and odd. More pairs may be eliminated if a certain even-odd digit combination does not exist on the list.

After further deductions, the following solutions were found:

$$
\left[\begin{array}{lllll}
1 & 8 & 2 & 2 & 5 \\
8 & 1 & 7 & 9 & 6 \\
2 & 7 & 5 & 5 & 6 \\
2 & 9 & 5 & 8 & 4 \\
5 & 6 & 6 & 4 & 4
\end{array}\right],\left[\begin{array}{lllll}
3 & 4 & 2 & 2 & 5 \\
4 & 5 & 7 & 9 & 6 \\
2 & 7 & 5 & 5 & 6 \\
2 & 9 & 5 & 8 & 4 \\
5 & 6 & 6 & 4 & 4
\end{array}\right],\left[\begin{array}{lllll}
5 & 7 & 1 & 2 & 1 \\
7 & 2 & 3 & 6 & 1 \\
1 & 3 & 9 & 2 & 4 \\
2 & 6 & 2 & 4 & 4 \\
1 & 1 & 4 & 4 & 9
\end{array}\right],\left[\begin{array}{lllll}
4 & 5 & 5 & 2 & 1 \\
5 & 3 & 3 & 6 & 1 \\
5 & 3 & 8 & 2 & 4 \\
2 & 6 & 2 & 4 & 4 \\
1 & 1 & 4 & 4 & 9
\end{array}\right],\left[\begin{array}{lllll}
2 & 5 & 9 & 2 & 1 \\
5 & 8 & 5 & 6 & 4 \\
9 & 5 & 4 & 8 & 1 \\
2 & 6 & 8 & 9 & 6 \\
1 & 4 & 1 & 6 & 1
\end{array}\right]
$$

So far, all solutions found have been symmetric matrices.

The following 16 symmetrical results were found by Jason Brown:

| 1 | 8 | 2 | 2 | 5 | 1 | 7 | 4 | 2 | 4 |  | 2 | 5 | 9 | 2 | 1 |  | 3 | 8 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 6 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 8 | 1 | 7 | 9 | 6 |  | 7 | 3 | 9 | 8 | 4 | 5 | 8 | 5 | 6 | 4 |  | 8 | 7 | 6 |
| 1 | 6 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 2 | 7 | 5 | 5 | 6 |  | 4 | 9 | 7 | 2 | 9 |  | 9 | 5 | 4 | 8 | 1 |  | 4 | 6 |
| 2 | 2 | 5 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 2 | 9 | 5 | 8 | 4 | 2 | 8 | 2 | 2 | 4 |  | 2 | 6 | 8 | 9 | 6 |  | 1 | 1 | 2 |
| 3 | 6 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 5 | 6 | 6 | 4 | 4 | 4 | 4 | 9 | 4 | 4 |  | 1 | 4 | 1 | 6 | 1 |  | 6 | 6 | 5 |
| 6 | 4 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 4 | 2 | 4 | 3 | 6 | 4 | 3 | 2 | 6 | 4 | 3 | 4 | 2 | 2 | 5 | 2 | 9 | 2 | 4 | 1 |
| 2 | 7 | 8 | 8 | 9 |  | 3 | 1 | 6 | 8 | 4 |  | 4 | 5 | 7 | 9 | 6 |  | 9 | 4 |
| 8 | 6 | 4 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 4 | 8 | 8 | 4 | 1 | 2 | 6 | 5 | 6 | 9 | 2 | 7 | 5 | 5 | 6 | 2 | 8 | 5 | 6 | 1 |
| 3 | 8 | 4 | 1 | 6 | 6 | 8 | 6 | 4 | 4 | 2 | 9 | 5 | 8 | 4 | 4 | 6 | 6 | 5 | 6 |
| 6 | 9 | 1 | 6 | 9 | 4 | 4 | 9 | 4 | 4 | 5 | 6 | 6 | 4 | 4 | 1 | 4 | 1 | 6 | 1 |
| 3 | 3 | 1 | 2 | 4 |  | 6 | 5 | 5 | 3 | 6 | 5 | 7 | 1 | 2 | 1 | 6 | 8 | 1 | 2 |
| 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 3 | 1 | 6 | 8 | 4 | 5 | 4 | 2 | 8 | 9 | 7 | 2 | 3 | 6 | 1 | 8 | 1 | 7 | 9 | 6 |
| 1 | 6 | 1 | 2 | 9 | 5 | 2 | 4 | 4 | 1 | 1 | 3 | 9 | 2 | 4 | 1 | 7 | 9 | 5 | 6 |
| 2 | 8 | 2 | 2 | 4 | 3 | 8 | 4 | 1 | 6 | 2 | 6 | 2 | 4 | 4 | 2 | 9 | 5 | 8 | 4 |
| 4 | 4 | 9 | 4 | 4 | 6 | 9 | 1 | 6 | 9 | 1 | 1 | 4 | 4 | 9 | 1 | 6 | 6 | 4 | 1 |
| 7 | 9 | 5 | 2 | 4 | 9 | 2 | 4 | 1 | 6 | 9 | 4 | 8 | 6 | 4 | 1 | 4 | 6 | 4 | 1 |
| 9 | 5 | 4 | 8 | 1 | 2 | 7 | 8 | 8 | 9 | 4 | 9 | 2 | 8 | 4 | 4 | 4 | 9 | 4 | 4 |
| 5 | 4 | 7 | 5 | 6 | 4 | 8 | 8 | 4 | 1 | 8 | 2 | 3 | 6 | 9 | 6 | 9 | 6 | 9 | 6 |
| 2 | 8 | 5 | 6 | 1 | 1 | 8 | 4 | 9 | 6 | 6 | 8 | 6 | 4 | 4 | 4 | 4 | 9 | 4 | 4 |
| 4 | 1 | 6 | 1 | 6 | 6 | 9 | 1 | 6 | 9 | 4 | 4 | 9 | 4 | 4 | 1 | 4 | 6 | 4 | 1 |

As a way to verify the accuracy and completeness of his program, a similar version found the unique solution found for the $4 \times 4 \mathrm{grid}$; it also produced the four solutions for a $2 \times 2$ grid, which can also be easily verified by hand.

## GRE Practice \#3: From ETS GRE Mathematics Practice Book

One of the greatest challenges students have with the math GRE subject test is that while they solve a problem, often it is faster to eliminate four wrong answers than find the exact
solution (or at least eliminate a few answers, at which point on average it is advantageous to guess). Consider the following (a discussion of the answer is included after the solutions to earlier PME problems), taken from one of the on-line collections of GRE problems (it was Problem 3). Find

$$
\int_{e^{-3}}^{e^{-2}} \frac{d x}{x \log x}
$$

The choices are (a) 1 , (b) $2 / 3$, (c) $3 / 2, \quad$ (d) $\log (2 / 3), \quad$ (e) $\log (3 / 2)$.

Solution: We can solve directly if we can find the anti-derivative of the integrand. As there are exponentials and logarithms and inverses, it's not unreasonable to expect a logarithm or exponential might play a part in the anti-derivative of $1 /(x \log x)$. If we do a $u$-substitution and set $u=\log x$ then $d u=d x / x$, the bounds of integration $x: e^{-3} \rightarrow e^{-2}$ becomes $u:-3 \rightarrow-2$ and we find

$$
\int_{x=e^{-3}}^{e^{-2}} \frac{d x}{x \log x}=\int_{u=-3}^{-2} \frac{d u}{u}=\left.\log (u)\right|_{u=-3} ^{-2}
$$

but this makes no sense as we cannot take a logarithm at a negative value!
The clock is ticking. What did we do wrong? The problem is we're integrating in a negative region. We should first change variables and let $w=-u$ and then integrate. So if $w=-u$ then $u:-3 \rightarrow-2$ is the same as $w: 3 \rightarrow 2$, and $d u / u=d w / w$ as the negative signs cancel. We find

$$
\int_{u=-3}^{-2} \frac{d u}{u}=\int_{w=3}^{2} \frac{d w}{w}=\left.\log (w)\right|_{w=3} ^{2}=\log (2)-\log (3)=\log (2 / 3)
$$

This is the correct answer, but it took us precious time to find it, we needed to see a $u$-substitution and we had an issue where we were evaluating a logarithm at a negative value (although, interestingly, if we did that and got $\log (-2)-\log (-3)$ and then used the quotient-difference rule to simplify this as $\log \left(\frac{-2}{-3}\right)$ we would find the right answer!).

Fortunately, there is a faster way to do this problem. Note this is not the same as there is a faster way to find the answer; we only need to eliminate four of the five answers. Let's try and get a sense of how large the integrand $1 /(x \log x)$ is with $x$ ranging from $e^{-3}$ to $e^{-2}$. At the two extremes we get

$$
\frac{1}{e^{-3} \log \left(e^{-3}\right)}=-\frac{1}{3 e^{-3}}, \quad \frac{1}{e^{-2} \log \left(e^{-2}\right)}=-\frac{1}{2 e^{-2}}
$$

This is enough to eliminate four of the answers! The reason is we see our integrand is negative throughout the entire region, and of the five answers only (d) $\log (2 / 3)$ is negative!

It's worth emphasizing that often the hardest part of this exam is time management. You want to get through the easier problems quickly to save time for the harder ones later on. We were fortunate here with the answers; only one was negative and the integrand was negative.

You really want to get in the mindset of looking quickly at the problem and getting a rough estimate of the solution. Frequently that is enough, as only one answer is often close....

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