PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2019

STEVEN J. MILLER (EDITOR)

1. Problems: Fall 2019

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. This issue's winner is the Missouri State University Problem Solving Group. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). After the Spring 2019 issue went to press we received a correct solution for #1351 from the Cal Poly Pomona Problem Solving Group.

This is a special themed issue, where all of the proposed problems are joint with my colleague and co-author Ron Evans.

#1362: *Proposed by Ron Evans (UCSD) and Steven J. Miller (Williams College).*

A weighted penny and a weighted dime are each flipped k times, for fixed $k \geq 4$. The penny has a probability p of landing heads, and the dime has a probability d of landing

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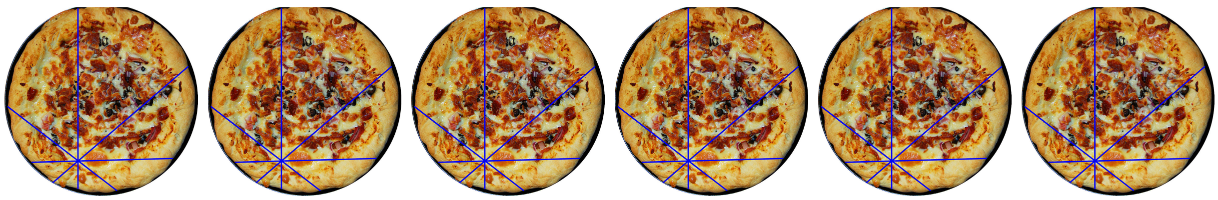


FIGURE 1. Pizza motivation; can you name the theorem that's represented here?

heads, where $.5 < p < d < 1$. Thus the expected numbers of heads obtained for the penny and dime after the flips are kp and kd, respectively.

Assuming that both expected values kp and kd are integers, prove that the probability of obtaining exactly kd heads for the dime exceeds the probability of obtaining exactly kp heads for the penny.

#1363: *Proposed by Ron Evans (UCSD) and Steven J. Miller (Williams College).*
This problem generalizes #1362, and starts off with the same setting. A weighted penny and a weighted dime are each flipped k times, for fixed $k \geq 4$. The penny has a probability p of landing heads, and the dime has a probability d of landing heads, where $.5 < p < d < 1$. Thus the expected numbers of heads obtained for the penny and dime after the flips are kp and kd, respectively.

We no longer assume kp and kd are integers. Thus, prove that when $d - p \geq 1/k$, the probability of obtaining exactly $r(kd)$ heads for the dime exceeds the probability of obtaining exactly $r(kp)$ heads for the penny, where $r(x)$ denotes the closest integer to x (rounding upwards when x is half an odd integer).

#1364: *Proposed by Ron Evans (UCSD) and Steven J. Miller (Williams College).*
(a) Two people are playing pool, which has 15 balls labeled from 1 to 15. On the first turn someone sinks three of these. What is the probability that none of the three numbers of the sunk balls are adjacent? Thus if they sink the 4, 8 and 12 that would work, but if they sank the 4, 8 and 9 it would not.
(b) Assume now they play super-pool, where there are 2020 balls on the table, and now someone sinks 314 balls on their first turn. What is the probability none of the 314 sunk have adjacent numbers?
(Bonus: More generally, what would the answer be if there were N balls and they sink k on their first turn?)

#1365: *Proposed by Ron Evans (UCSD) and Steven J. Miller (Williams College).*
We generalize the previous problem. We have N balls, numbered 1 to N.
(a) Assume exactly k balls are sunk on the first turn. What is the probability that there are no three consecutive numbers among them?
(b) Assume exactly k balls are sunk on the first turn, so there are $N - k$ that are not sunk. What is the probability that there are no three consecutive numbers among the k sunk and also there are no three consecutive numbers among the $N - k$ not sunk?

GRE Practice #5: From a GRE Practice Exam: Let a, b be positive; determine

$$\int_0^\infty \frac{\exp(ax) - \exp(bx)}{(1+\exp(ax))\cdot(1+\exp(bx))}dx.$$

(a) 0 (b) 1 (c) $a-b$ (d) $(a-b)\log(2)$ (e) $\frac{a-b}{ab}\log(2)$.

2. Solutions

#1302: *Proposed by Steven Finch, Harvard University, Cambridge, MA.* Let A, B, C, D be independent uniform random points on the unit sphere Σ in $\mathbb{R}^3$. The points A, B, C determine a unique disk Δ inscribed within Σ almost surely. Let Γ denote the oblique circular cone with base Δ and apex D. The volume ω of Γ cannot exceed $32\pi/81$. Find the probability density function for ω in closed-form. Find the first and second moments of ω as well. Note: The density function here is, in fact, algebraic in ω! This is believed to be rare for such problems in geometric probability.

Solution by proposer: Assume without loss of generality that the plane containing Δ in xyz-space is parallel to the plane $z=0$. Let θ denote the angular radius of Δ relative to the north pole $E=(0,0,1)$; in particular, θ is the angle between vectors A and E. For example, Δ is the equatorial disk iff $\theta=\pi/2$; $\Delta=\{E\}$ iff $\theta=0$; and $\Delta=\{-E\}$ iff $\theta=\pi$.

Assume further WLOG that point D is in the northern hemisphere $z>0$. Given θ, the maximum volume $\omega=\frac{\pi}{3}\sin^2\theta(1-\cos\theta)$ occurs when $D=E$. Given ω, the largest interval $\theta_0 \le \theta \le \theta_1$ such that $\omega \le \frac{\pi}{3}\sin^2\theta(1-\cos\theta)$ has endpoints

$$\theta_0 = \arccos\zeta_0, \qquad \theta_1 = \arccos\zeta_1$$

where $-1 \le \zeta_1 \le \zeta_0 \le 1$ satisfy uniquely the cubic equation

$$\omega = \frac{\pi}{3}\left(1-\zeta^2\right)(1-\zeta).$$

Expressions for ζ_0, ζ_1 are complicated and hence omitted.

Given θ, it is clear from [1] that $\omega \sim \text{Uniform}\left[0, \frac{\pi}{3}\sin^2\theta(1-\cos\theta)\right]$. Thus the conditional density of $\omega|\theta$ is simple. It is more challenging to show that the unconditional density of θ is

$$\frac{3}{4}\sin^3\theta\,(1-\cos\theta), \qquad 0 \le \theta \le \pi$$

and required details appear in [2] (proof of Theorem 3.3). Integrating the product of the two densities, we obtain

$$\frac{3}{4}\frac{3}{\pi}\int\limits_{\theta_0(\omega)}^{\theta_1(\omega)} \sin\theta\,d\theta = -\frac{i\,3^{3/2}\left(-8\pi^{2/3} + 2^{1/3}\left\{-16\pi + 9\left[9\omega + i\sqrt{(32\pi-8\omega)\omega}\right]\right\}^{2/3}\right)}{2^{8/3}\pi^{4/3}\left\{-16\pi + 9\left[9\omega + i\sqrt{(32\pi-8\omega)\omega}\right]\right\}^{1/3}}$$

as the density of ω. From this, $\mathbb{E}(\omega)=16\pi/105$ and $\mathbb{E}(\omega^2)=32\pi^2/945$ follow immediately.

Notes: An alternative proof of Miles' result [2] can be found in [3]. The points A, B, C, D also determine a random tetrahedron with mean volume $4\pi/105$ and mean square volume $2/81$ [4, 5]. Similar reasoning gives the area density for random triangles inscribing the unit circle in $\mathbb{R}^2$, but elementary evaluation of the corresponding integral seems unlikely [6].

References

[1] M. Horowitz, Probability of random paths across elementary geometrical shapes, *J. Appl. Probab.* 2 (1965) 169–177; MR0198509 (33 #6664).
[2] R. E. Miles, Random points, sets and tessellations on the surface of a sphere, *Sankhya Ser. A* 33 (1971) 145–174; MR0321150 (47 #9683).
[3] P. Bürgisser, F. Cucker and M. Lotz, Coverage processes on spheres and condition numbers for linear programming, *Annals of Probab.* 38 (2010) 570–604; MR2642886 (2011d:60036).
[4] R. E. Miles, Isotropic random simplices, *Adv. Appl. Probab.* 3 (1971) 353–382; MR0309164 (46 #8274).
[5] L. Heinrich, R. Körner, N. Mehlhorn and L. Muche, Numerical and analytical computation of some second-order characteristics of spatial Poisson-Voronoi tessellations, *Statistics* 31 (1998) 235–259; MR1718316 (2000g:60016).
[6] A. M. Mathai and D. S. Tracy, On a random convex hull in an n-ball, *Comm. Statist. A – Theory Methods* 12 (1983) 1727–1736; MR0704849 (85c:60013).

#1356: *Proposed by Greg Oman and Ikko Saito, University of Colorado, Colorado Springs.*
Problem. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$. For $r \in \mathbb{R}$, say that f is *homomorphic at* r if $f(r + x) = f(r) + f(x)$ for all $x \in \mathbb{R}$. Next, set $\mathcal{H}_f := \{r \in \mathbb{R}\colon f$ is homomorphic at $r\}$. One can check that $\mathcal{H}_f$ is an additive subgroup of $\mathbb{R}$ (which may be assumed in your solution). For the purposes of this problem, say that a subgroup G of $\mathbb{R}$ is *realizable* if $G = \mathcal{H}_f$ for some continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$.

(a) Prove that every (additive) cyclic subgroup of $\mathbb{R}$ is realizable.
(b) Find all non-cyclic realizable subgroups of $\mathbb{R}$.

Solution by ***Missouri State University Problem Solving Group, Missouri State University***.

We first show that every subgroup of $\mathbb{R}$ is either cyclic or is dense in $\mathbb{R}$.

Let G be any subgroup of $\mathbb{R}$. If $G = \{0\}$, then G is cyclic. Otherwise G contains positive elements, i.e., $G \cap (0, \infty) \neq \emptyset$. Let $t = \inf(G \cap (0, \infty))$.

Case 1: $t = 0$. Then for every $\epsilon > 0$ there exists $g \in G$ such that $0 < g \leq \epsilon$. We show that G is dense by showing that $G \cap (a, b) \neq \emptyset$ for every open interval $(a, b) \subseteq \mathbb{R}$. Given an open interval (a, b), take $\epsilon = (b-a)/2$. Then there exists $g \in G$ such that $0 < g \leq \epsilon < b-a$. Since g is smaller than the length of (a, b), for some integer k we have $kg \in (a, b)$, and also $kg \in G$ since G is a group. This shows that $G \cap (a, b)$ is nonempty. Therefore G is dense in $\mathbb{R}$.

Case 2: $t > 0$. In this case we show $G = t\mathbb{Z}$. First, we have $t \in G$ for the following reason. Suppose $t \notin G$. Because $t = \inf\{g \in G : g > 0\}$, for any $\epsilon > 0$ we can find $g \in G$ such that $t < g < t + \epsilon$. Also, we can find $g' \in G$ such that $t < g' < g < t + \epsilon$. Thus $0 < g - g' < \epsilon$ and $g - g' \in G$. Choosing ϵ smaller than t gives a contradiction and therefore $t \in G$. Then $t\mathbb{Z} \subseteq G$. Let $g \in G$. Choose $k \in \mathbb{Z}$ such that $kt \leq |g| < (k+1)t$. Now, $|g| - kt \in G$, and $0 \leq |g| - kt < t$. By definition of t, $|g| - kt = 0$. That is, $g = \pm kt \in t\mathbb{Z}$.

(a) Let $G = t\mathbb{Z}$ be any (additive) cyclic subgroup of $\mathbb{R}$. Consider the function $f(x) = \sin(2\pi x/t)$, a continuous function $f : \mathbb{R} \to \mathbb{R}$ with period t. We have

$$f(tn + x) = \sin(2\pi(tn + x)/t) = \sin(2\pi n + 2\pi x/t) = \sin(2\pi x/t) = f(x)$$

for every integer n, and it follows that $G = \mathcal{H}_f$.

(b) Suppose G is a non-cyclic realizable subgroup of $\mathbb{R}$. Then $G = \mathcal{H}_f$ is dense in $\mathbb{R}$. Let $s \in \mathbb{R}$. There exists a sequence $r_n \in \mathcal{H}_f$ such that $\lim_{n\to\infty} r_n = s$. Therefore,

$$\begin{aligned} f(s+x) = f(\lim_{n\to\infty}(r_n+x)) &= \lim_{n\to\infty} f(r_n+x) \\ &= \lim_{n\to\infty} f(r_n) + f(x) \\ &= f(\lim_{n\to\infty} r_n) + f(x) \;=\; f(s)+f(x) \end{aligned}$$

which shows f is homomorphic at every real number s. This shows $\mathcal{H}_f = \mathbb{R}$, and the only non-cyclic realizable subgroup is $\mathbb{R}$.

#1357: *Proposed by Ron Evans (UCSD) and Steven J. Miller (Williams).*

Let n be a positive integer. A pin of length n units is dropped randomly onto a large floor ruled with equally spaced parallel lines 1 unit apart. When it lands, the pin can intersect k parallel lines, where k is an integer between 0 and $n+1$ inclusive. If the center of the pin lands halfway between two adjacent lines, which value of k is most probable?

Solution by ***Skidmore Problem Group, Skidmore College****.*

The solution is

$$k \;=\; 2\left\lfloor \frac{n}{2} \right\rfloor.$$

Let us call the family of equally spaced parallel lines $\mathcal{F}$. Let C be the point on the floor under the center of the pin, and let Γ be the circle centered at C with diameter n. Note that the number of intersections of the pin with $\mathcal{F}$ is even, since C lies halfway between two adjacent elements. So if n is even, $0 \le k \le n$ and if n is odd, $0 \le k \le n+1$. Let S_0 be the intersection of the line through C parallel to the elements of $\mathcal{F}$ with Γ, and let $S_1, S_2, ..., S_r$ be the sequence of intersections with $\mathcal{F}$ moving along Γ such that $\angle S_0CS_m < \frac{\pi}{2}$. Let S_{r+1} be the first intersection of Γ with the line through C perpendicular to the elements of $\mathcal{F}$. Note that for every $n, r = \left\lfloor \frac{n}{2} \right\rfloor$. Note also that every is pin dropped so that an endpoint lies between S_i and S_{i+1} has $k = 2i$ intersections for $i = 0, 1, \ldots, r$. Due to the symmetry of Γ, it will suffice to seek the circular sectors with arcs between S_i and S_{i+1} which have the most possible pin positions. But these positions are determined by their angles with $\overline{\mathrm{CS}_0}$. So let the measure of $\angle S_0CS_i = \alpha_i$. We wish to maximize $\alpha_{i+1} - \alpha_i$, where $i = 0, 1, \ldots, r$, which will maximize the probability. Now for each of these values of i,

$$\sin(\alpha_i) \;=\; \frac{i - \frac{1}{2}}{\frac{n}{2}} \;=\; \frac{2i-1}{n}.$$

So,

$$\alpha_i \;=\; \sin^{-1}\left(\frac{2i-1}{n}\right),$$

which gives,

$$\alpha_{i+1} - \alpha_i \;=\; \sin^{-1}\left(\frac{2i+1}{n}\right) - \sin^{-1}\left(\frac{2i-1}{n}\right).$$

Let us say

$$\alpha_{i+1} - \alpha_i \;=\; f(i).$$

Then

$$\begin{aligned} f'(i) &= \frac{2}{n\sqrt{1-\left(\frac{2i+1}{n}\right)^2}} - \frac{2}{n\sqrt{1-\left(\frac{2i-1}{n}\right)^2}} \\ &= \frac{2}{n}\left(\frac{1}{\sqrt{1-\left(\frac{2i+1}{n}\right)^2}} - \frac{1}{\sqrt{1-\left(\frac{2i-1}{n}\right)^2}}\right). \end{aligned}$$

However,

$$\left(\frac{2i+1}{n}\right)^2 > \left(\frac{2i-1}{n}\right)^2,$$

which implies

$$\sqrt{1-\left(\frac{2i+1}{n}\right)^2} < \sqrt{1-\left(\frac{2i-1}{n}\right)^2},$$

giving

$$\frac{2}{n\sqrt{1-\left(\frac{2i+1}{n}\right)^2}} > \frac{2}{n\sqrt{1-\left(\frac{2i-1}{n}\right)^2}}.$$

This means $f'(i) > 0$, so f is an increasing function which attains its maximum at $i = r$. Therefore, the maximal value of $f(i)$ occurs at $i = r$, that is, $k = 2\left\lfloor \frac{n}{2} \right\rfloor$.

#1359: *Proposed by Robert C. Gebhardt, Chester, NJ.* Determine the following sums:

$$\begin{aligned} (a) &\qquad \frac{1}{1+2} - \frac{1}{3+4} + \frac{1}{5+6} - \frac{1}{7+8} + \cdots \\ (b) &\qquad \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \cdots \\ (c) &\qquad \frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \cdots \end{aligned}$$

Solution by ***Hongwei Chen, Christopher Newport University****. Also solved by* ***Brian Bradie, Christopher Newport University****,* ***Eugen Ionascu, Columbus State University****,* ***Ioannis D. Sfikas, Athens, Greece****, and* ***Kenneth Davenport****.*

Solution. (a). The value is $\frac{\sqrt{2}}{8}(\pi + \ln(3-2\sqrt{2}))$. Notice that the series can be rewritten as

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{11} - \frac{1}{15} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n-1}.$$

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{4n-1}}{4n-1}, \qquad |x| < 1.$$

Since the above power series has the radius of convergence of 1, term-wise differentiating yields

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1}x^{4n-2} = \frac{x^2}{1+x^4}, \qquad |x| < 1.$$

By the Abel theorem, we find that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n-1} \;=\; f(1) \;=\; \int_0^1 f'(x)\,dx \;=\; \int_0^1 \frac{x^2\,dx}{1+x^4}.$$

Mathematica gives the claimed answer directly. Here we provide an elementary evaluation:

$$\begin{aligned}
\int_0^1 \frac{x^2\,dx}{1+x^4} &= \frac{1}{2}\int_0^1 \frac{(x^2+1)+(x^2-1)}{1+x^4}\,dx \\
&= \frac{1}{2}\left(\int_0^1 \frac{x^2+1}{1+x^4}\,dx + \int_0^1 \frac{x^2-1}{1+x^4}\,dx\right) \\
&= \frac{1}{2}\left(\int_0^1 \frac{1+1/x^2}{(x-1/x)^2+2}\,dx + \int_0^1 \frac{1-1/x^2}{(x+1/x)^2-2}\,dx\right) \\
&= \frac{1}{2}\left(\int_{-\infty}^0 \frac{du}{u^2+2} - \int_2^{\infty} \frac{dv}{v^2-2}\right) \qquad (\text{let } u = x-1/x, v = x+1/x) \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\arctan^{-1}\left(\frac{u}{\sqrt{2}}\right)\Big|_{-\infty}^{0} - \frac{1}{2\sqrt{2}}\ln\left(\frac{v-\sqrt{2}}{v+\sqrt{2}}\right)\Big|_2^{\infty}\right) \\
&= \frac{\sqrt{2}}{8}(\pi + \ln(3-2\sqrt{2})),
\end{aligned}$$

where partial fractions is used to evaluate the second integral.

(b). The value is $\ln 2$. Rewrite the series as $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)}$. Let s_n be the n-partial sum of the series. Partial fractions yields

$$s_n \;=\; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$

It is well-known that the above sequence converges to $\ln 2$.

(c). The value is $\frac{\pi}{4} - \frac{1}{2}\ln 2$. Similar to (b), let S_n be the n-partial sum of the series. Partial fractions and rearranging the terms yield

$$\begin{aligned}
S_n &= 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \cdots + (-1)^{n+1}\left(\frac{1}{2n-1} - \frac{1}{2n}\right) \\
&= \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n+1}\frac{1}{2n-1}\right) \\
&\quad -\frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1}\frac{1}{n}\right).
\end{aligned}$$

Invoking the well-known numerical series for $\pi/4$ and $\ln 2$, we find that

$$S_n \to \frac{\pi}{4} - \frac{1}{2}\ln 2 \quad \text{as } n \to \infty.$$

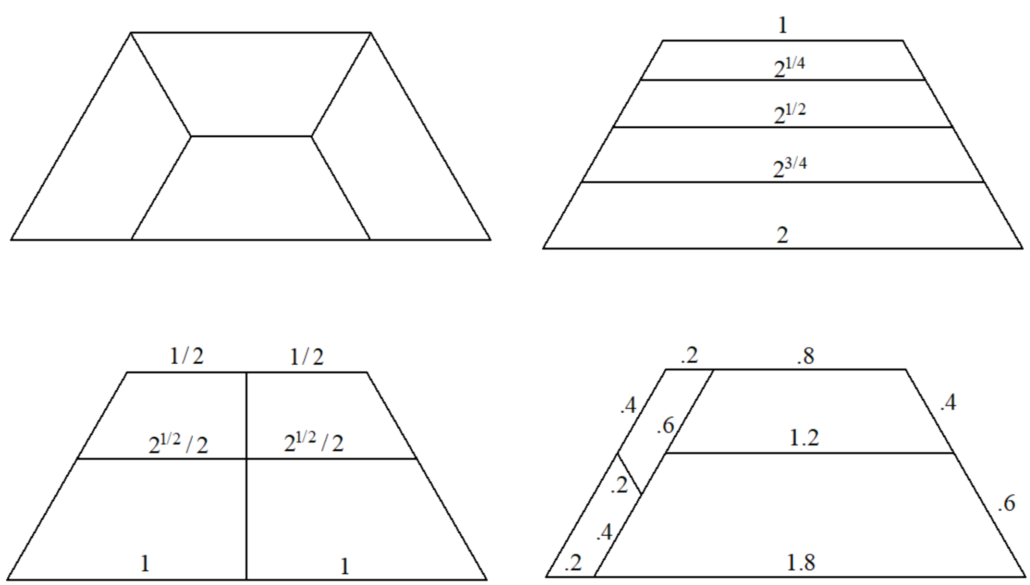
#1360: *Proposed by Stanley Wu-Wei Liu, East Setauket, Long Island, New York.*

Cutting a cake, be it round or otherwise, is a fun skill with real-world applications. When mathematicians work on such dissection problems starting out with a *quadrilateral*-shaped cake and whimsically demanding that the constituent pieces be *similar polygons* (in the precise Euclidean sense), a lot is known when the number of these similar polygons is chosen to be four. Consider the case of an **isosceles trapezoid with side-length ratios of 1:1:1:2**. There are many fascinating solutions; find at least four partitions of the 1:1:1:2 isosceles trapezoid into four similar polygons.

The inspiration for this problem came from an MIT Puzzle Corner problem (edited by Professor Allan Gottlieb, NYU) appearing in the July / August 2018 issue of *MIT News*, in which Problem J/A 2 (on page 63) is stated as follows:

> **J/A 2**. Dick Hess had sent us the following problem, which he attributed to Bob Wainwright. The diagram below shows an equilateral [sic] trapezoid constructed from three equilateral triangles. You are to divide the figure into four similar pieces of three different sizes (i.e., exactly two pieces are congruent).

See also https://s3.amazonaws.com/files.technologyreview.com/p/pub/magazine/mitnews/puzzlecorner/JA18MITPuzzleCorner.pdf, and for more on this problem, including readings, generalizations and open problems, email the proposer at swwliu@alum.mit.edu.
Solution by ***Missouri State University Problem Solving Group, Missouri State University****.*



GRE Practice #5: *Solution by Steven Miller, Williams College*
From a GRE Practice Exam: Let a, b be positive; determine

$$\int_0^\infty \frac{\exp(ax) - \exp(bx)}{(1+\exp(ax)) \cdot (1+\exp(bx))} dx.$$

(a) 0 (b) 1 (c) $a-b$ (d) $(a-b)\log(2)$ (e) $\frac{a-b}{ab}\log(2)$.

Solution: As the integral vanishes when $a = b$ and is positive when $a > b$, we can eliminate the first two (there's also a huge hint from the fact that three of the five answers have a factor of $a - b$). For the last three it requires a bit of work.

One approach is to use units. If x is in meters, then a and b must be in 1/meters, and the integrand will be in meters. That suggests the fifth answer, (e), as that's the only combination of the last three that will have units of meters.

Here's another way to see this: check extreme values. We know it's positive if $a > b$; let's send a to infinity and b to zero. If we do that, the numerator is essentially $\exp(ax)$, as that dwarfs the $\exp(bx)$.

However, there's a lot more that we can see. Note that for a large, if we group $\exp(ax)/(1+\exp(ax))$ we see that factor is essentially 1. Thus the integral looks a lot like the integral of $1/(1 + \exp(bx))$. Actually, we don't need to send b to zero, we can have b large if we want, just much smaller than a so the numerator is dominated by the first term. In that case, the integrand looks a lot like $\exp(-bx)$, and if we integrate that from zero to infinity we get $1/b$.

So, to recap, if a and b are tending to infinity and a is much larger than b, then the integral should look like $1/b$. Now we win, as only one of the five answers looks like that, the fifth, as $(a - b)/(ab) = 1/b - 1/a$. (Note, as a sanity check, that since a is much larger than b, then $1/b - 1/a$ is going to be positive, as it should be.)

Thus we are able to eliminate four of the five answers without finding the anti-derivative, without evaluating the integral! This demonstrates yet again the power of stepping back and remembering the wisdom of Sherlock Holmes: if you eliminate the impossible, whatever remains, however improbable, must be true.

Of course, these are just my thoughts on how to tackle this problem. My colleague Ron Evans has two more. First, let $J(a, b)$ denote the value of the integral. Substituting x/ab for x in the integral, we see that $J(a, b) = J(1/b, 1/a)/ab$. This rules out choices (b), (c), and (d). Since we know the answer is not 0, that leaves only choice (e).

Alternatively, replace the numerator $\exp(ax)-\exp(bx)$ by $(1+\exp(ax))-(1+\exp(bx))$, and now the integral breaks up into easily evaluated integrals via substitution. This approach is useful in cases where the GRE has "none of the above" as one of the multiple choices.

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