# PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2019

STEVEN J. MILLER (EDITOR)

## 1. Problems: Spring 2019

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published. Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to \$100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). This issue's winning school is *North Central College*; congratulations!

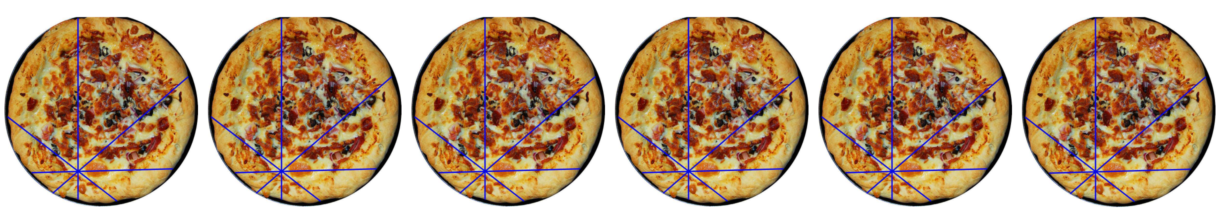

Figure 1. Pizza motivation; can you name the theorem that's represented here?

*Date*: April 10, 2019.

*Note:* After the Fall 2018 pages went to press, several additional solutions were received. These include #1348 by Kenneth Davenport.

**#1356:** *Proposed by Greg Oman and Ikko Saito, University of Colorado, Colorado Springs.*
**Problem.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$. For $r \in \mathbb{R}$, say that $f$ is *homomorphic at* $r$ if $f(r + x) = f(r) + f(x)$ for all $x \in \mathbb{R}$. Next, set $\mathcal{H}_f := \{r \in \mathbb{R}\colon f$ is homomorphic at $r\}$. One can check that $\mathcal{H}_f$ is an additive subgroup of $\mathbb{R}$ (which may be assumed in your solution). For the purposes of this problem, say that a subgroup $G$ of $\mathbb{R}$ is *realizable* if $G = \mathcal{H}_f$ for some continuous $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$.

(a) Prove that every (additive) cyclic subgroup of $\mathbb{R}$ is realizable.
(b) Find all non-cyclic realizable subgroups of $\mathbb{R}$.

**#1357:** *Proposed by Ron Evans (UCSD) and Steven J. Miller (Williams).*
Let $n$ be a positive integer. A pin of length $n$ units is dropped randomly onto a large floor ruled with equally spaced parallel lines 1 unit apart. When it lands, the pin can intersect $k$ parallel lines, where $k$ is an integer between 0 and $n + 1$ inclusive. If the center of the pin lands halfway between two adjacent lines, which value of $k$ is most probable?

**#1358:** *Proposed by Ron Evans (UCSD) and Steven J. Miller (Williams).*
Let $n$ be a positive integer. A pin of length $n$ units is dropped randomly onto a large floor ruled with equally spaced parallel lines 1 unit apart. When it lands, the pin can intersect $k$ parallel lines, where $k$ is an integer between 0 and $n + 1$ inclusive. Which value of $k$ is most probable? (Note unlike the previous problem, now there is no restriction on the location of the center.)

**#1359:** *Proposed by Robert C. Gebhardt, Chester, NJ.*
Determine the following sums:

$$(a) \qquad \frac{1}{1+2} - \frac{1}{3+4} + \frac{1}{5+6} - \frac{1}{7+8} + \cdots$$
$$(b) \qquad \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \cdots$$
$$(c) \qquad \frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} - \frac{1}{7\cdot 8} + \cdots$$

**#1360:** *Proposed by Stanley Wu-Wei Liu, East Setauket, Long Island, New York.*
Cutting a cake, be it round or otherwise, is a fun skill with real-world applications. When mathematicians work on such dissection problems starting out with a *quadrilateral*-shaped cake and whimsically demanding that the constituent pieces be *similar polygons* (in the precise Euclidean sense), a lot is known when the number of these similar polygons is chosen to be four. Consider the case of an **isosceles trapezoid with side-length ratios of 1:1:1:2**. There are many fascinating solutions; find at least four partitions of the 1:1:1:2 isosceles trapezoid into four similar polygons.

The inspiration for this problem came from an MIT Puzzle Corner problem (edited by Professor Allan Gottlieb, NYU) appearing in the July / August 2018 issue of *MIT News*, in which Problem J/A 2 (on page 63) is stated as follows:

> **J/A 2**. Dick Hess had sent us the following problem, which he attributed to Bob Wainwright. The diagram below shows an equilateral [sic] trapezoid constructed from three equilateral triangles. You are to divide the figure into four similar pieces of three different sizes (i.e., exactly two pieces are congruent).

See also https://s3.amazonaws.com/files.technologyreview.com/p/pub/magazine/mitnews/puzzlecorner/JA18MITPuzzleCorner.pdf, and for more on this problem, including readings, generalizations and open problems, email the proposer at swwliu@alum.mit.edu.

**#1361:** *Proposed by Clayton Mizgerd and Steven J. Miller, Williams College.*
Consider a circle of radius 2019. For each pair of positive integers $c$ and $k$, find a positive integer $N$ (which may depend on $c$ and $k$) such that if we choose any $k$ points on the circle, then there is at least one closed arc $\mathcal{A}$ such that the length of the arc $\mathcal{A}$ is $1/c$ of the length of the perimeter of the circle, and at least $k$ of the $N$ points are on that arc.

**GRE Practice #4:** *Proposed by Steven Miller, Williams College*
One of the greatest challenges students have with the math GRE subject test is that while they solve a problem, often it is faster to eliminate four wrong answers than find the exact solution (or at least eliminate a few answers, at which point on average it is advantageous to guess). Consider the following (a discussion of the answer is included after the solutions to earlier PME problems), taken from one of the on-line collections of GRE problems (it was Problem 45). How many positive numbers $x$ satisfy the equation $\cos(97x) = x$? The options are (a) 1, (b) 15, (c) 31, (d) 49, (e) 96.

**Special Bonus: The Four Four Game:** The Four Fours game (or problem) asks you to represent as many numbers as you can using exactly four fours (though some variants just require you to use at most four fours). For example,

$$1 = \frac{44}{44} \quad \text{or} \quad (4/4)^{4*4} \quad \text{or} \quad \sqrt[4]{4^4}/4,$$

to name a few. A colleague of mine, Steve Conrad of http://www.mathleague.com/, has collected some of his favorites. We stated the problem in the last issue (Fall 2019), and provide the promised expansions in Figure 2.

## 2. Solutions

**#1348:** *Proposed by Matthew McMullen, Otterbein University.*
An equable triangle is one whose area and perimeter evaluate to the same number. Find the real number $a$ such that there exists exactly one equable triangle with two sides of length $a$. (**Bonus:** Classify all pairs of real numbers $(a, b)$ with $a \geq b$ such that there exists exactly one equable triangle with one side of length $a$ and another side of length $b$.)
*Solution by **Ioannis D. Sfikas, Athens, Greece**.*

0. $4+4-4-4$
1. $\frac{4}{4}\times\frac{4}{4}$
2. $\frac{4}{4}+\frac{4}{4}$
3. $\frac{4+4+4}{4}$
4. $4+4-\sqrt{4}-\sqrt{4}$
5. $\frac{4\times 4+4}{4}$
6. $4+4-4+\sqrt{4}$
7. $4+4-\frac{4}{4}$
8. $4+4+4-4$
9. $4+4+\frac{4}{4}$
10. $4\times 4-4-\sqrt{4}$
11. $4+\frac{4!+4}{4}$
12. $4\times 4-\sqrt{4}-\sqrt{4}$
13. $\frac{4!+4!+4}{4}$
14. $4+4+4+\sqrt{4}$
15. $4\times 4-\frac{4}{4}$
16. $4+4+4+4$
17. $4\times 4+\frac{4}{4}$
18. $4\times 4+4-\sqrt{4}$
19. $4!-4-\frac{4}{4}$
20. $4\times\left(4+\frac{4}{4}\right)$
21. $4!-4+\frac{4}{4}$
22. $4\times 4+4+\sqrt{4}$
23. $4!-\sqrt{4}+\frac{4}{4}$
24. $4\times 4+4+4$
25. $4!+\sqrt{4}-\frac{4}{4}$
26. $4(4+\sqrt{4})+\sqrt{4}$
27. $4!+\sqrt{4}+\frac{4}{4}$
28. $4!+4+4-4$
29. $4!+4+\frac{4}{4}$
30. $\sqrt{4}\times 4\times 4-\sqrt{4}$
31. $4!+\sqrt{4}+\frac{\sqrt{4}}{.4}$
32. $4\times 4+4\times 4$
33. $4!+4+\frac{\sqrt{4}}{.4}$
34. $4\times 4\times\sqrt{4}+\sqrt{4}$
35. $4!+\frac{4!-\sqrt{4}}{\sqrt{4}}$
36. $4!+4+4+4$
37. $4!+\frac{4!+\sqrt{4}}{\sqrt{4}}$
38. $4!+4\times 4-\sqrt{4}$
39. $4!+\frac{4+\sqrt{4}}{.4}$
40. $4\times 4\times 4-4!$
41. $\frac{4\times 4+.4}{.4}$
42. $4!+4!-4-\sqrt{4}$
43. $\frac{4!-4}{.\overline{4}}-\sqrt{4}$
44. $4!+4!-\sqrt{4}-\sqrt{4}$
45. $\frac{4\times 4+\sqrt{4}}{.4}$
46. $4!+4!-4+\sqrt{4}$
47. $4!+4!-\frac{4}{4}$
48. $4(4+4+4)$
49. $4!+4!+\frac{4}{4}$
50. $4!+4!+4-\sqrt{4}$
51. $\frac{4!-\sqrt{4}}{.4}-4$
52. $4!+4!+\sqrt{4}+\sqrt{4}$
53. $4!+4!+\frac{\sqrt{4}}{.4}$
54. $4!+4!+4+\sqrt{4}$
55. $\frac{4!}{.4}-\frac{\sqrt{4}}{.4}$
56. $4!+4!+4+4$
57. $\frac{4!-\sqrt{4}}{.4}+\sqrt{4}$
58. $4!+4!+\frac{4}{.4}$
59. $\frac{4!}{.4}-\frac{4}{4}$
60. $4\times 4\times 4-4$
61. $\frac{4!}{.4}+\frac{4}{4}$
62. $4\times 4\times 4-\sqrt{4}$
63. $\frac{4^4-4}{4}$
64. $(4+4)(4+4)$
65. $\frac{4^4+4}{4}$
66. $4\times 4\times 4+\sqrt{4}$
67. $\sqrt{4}+\frac{4!+\sqrt{4}}{.4}$
68. $4\times 4\times 4+4$
69. $4+\frac{4!+\sqrt{4}}{.4}$
70. $4!+4!+4!-\sqrt{4}$
71. $\frac{4!+4+.4}{.4}$
72. $4(4\times 4+\sqrt{4})$
73. $\sqrt{\sqrt{\sqrt{4^{4!}}}}+\frac{4}{.\overline{4}}$
74. $4!+4!+4!+\sqrt{4}$
75. $\frac{4!+4+\sqrt{4}}{.4}$
76. $4!+4!+4!+4$
77. $\left(\frac{4}{.\overline{4}}\right)^{\sqrt{4}}-4$
78. $4(4!-4)-\sqrt{4}$
79. $\frac{4!-\sqrt{4}}{.4}+4!$
80. $4(4\times 4+4)$
81. $\left(4-\frac{4}{4}\right)^4$
82. $4(4!-4)+\sqrt{4}$
83. $\frac{4!-.4}{.4}+4!$
84. $4(4!-4)+4$
85. $\frac{4!+.4}{.4}+4!$
86. $4(4!-\sqrt{4})-\sqrt{4}$
87. $4\times 4!-\frac{4}{.\overline{4}}$
88. $4\times 4!-4-4$
89. $\frac{4!+\sqrt{4}}{.4}+4!$
90. $4\times 4!-4-\sqrt{4}$
91. $4\times 4!-\frac{\sqrt{4}}{.4}$
92. $4\times 4!-\sqrt{4}-\sqrt{4}$
93. $4\times 4!-\sqrt{\frac{4}{.\overline{4}}}$
94. $4\times 4!+\sqrt{4}-4$
95. $4\times 4!-\frac{4}{4}$
96. $4!+4!+4!+4!$
97. $4\times 4!+\frac{4}{4}$
98. $4\times 4!+4-\sqrt{4}$
99. $\frac{4!\times\sqrt{4}-4}{.\overline{4}}$
100. $4\left(4!+\frac{4}{4}\right)$

FIGURE 2. These 'simple' solutions were submitted to Steve Conrad by students primarily from the U.S. and Canada when he was the problem editor of the NCTM's *Mathematics Student Journal*. In addition to the standard operations $(+,-,\cdot,\backslash)$, students also used square-roots, overline for infinite decimal expansions, and factorials. Neither 44 nor 4.4 were used. The hardest was 73, though 77, 81, 83, 87, 89 and 93 were also challenging.

Let $(a,b,a)$ be the sides of the triangle. In the usual notation, we have

$$A \;=\; (s-a)\sqrt{s(s-b)} \;=\; \frac{b}{4}\sqrt{4a^2-b^2}, \quad P \;=\; 2s \;=\; 2a+b,$$

where $s = \frac{a+b+c}{2}$ is the semiperimeter, $A$ is the area and $P$ the perimeter of the triangle $(a,b,a)$. Since we have

$$s \;=\; \frac{2a+b}{2}, \quad s-a \;=\; \frac{b}{2}, \quad s-b \;=\; \frac{2a-b}{2},$$

we have from the hypothesis and the classical area formula of Heron that $A = P$ implies

$$(s-a)\sqrt{s(s-b)} \;=\; 2s \quad \text{or} \quad (s-a)\sqrt{s-b} \;=\; 2\sqrt{s},$$

which yields

$$2(b^2-16)a \;=\; (b^2+16)b.$$

If $b^2 = 16$, we have $0a = 32b$, or $b = 0$, a contradiction. So, we have $b \neq 4$, and

$$a \;=\; a(b) \;=\; \frac{(b^2+16)b}{2(b^2-16)}.$$

Since $a < 0$ for $0 < b < 4$,

$$\lim_{b\to\infty} a(b) \;=\; \lim_{b\to\infty} \frac{(b^2+16)b}{2(b^2-16)} \;=\; \infty$$

and

$$\lim_{b\to 2} a(b) \;=\; \lim_{b\to 2^+} \frac{(b^2+16)b}{2(b^2-16)} \;=\; \infty$$

then the function $a = a(b)$ is not $1-1$ for $a = a_{\text{min}}$. So, we can easily calculate that for $b = 4\sqrt{2+\sqrt{5}} \approx 8.2327$ we have $a_{\text{min}} = \frac{2\sqrt{2+\sqrt{5}(3+\sqrt{5}))}}{1+\sqrt{5}} \approx 6.6604$. We find

$$P \;=\; 2a+b \;=\; 2\sqrt{58+26\sqrt{5}} \;\approx\; 21.5534$$

and

$$A \;=\; \frac{b}{4}\sqrt{4a^2-b^2} \;=\; 2\sqrt{58+26\sqrt{5}} \;\approx\; 21.5534.$$

**Bonus solution (from the proposer).** Suppose that the three sides of a triangle have length $a$, $b$, and $2x$, where $a \geq b$. Put $u = \frac{a+b}{2}$ and $v = \frac{a-b}{2}$. Then $a = u+v$ and $b = u-v$. Note that $0 \leq v < x < u$ ($x$ must be between $v$ and $u$ due to the triangle inequality). By Heron's Formula, the area of this triangle is given by

$$A \;=\; \sqrt{(u+x)(u-x)(x-v)(x+v)}.$$

Our triangle is equable only if $2u + 2x = A$; or, after some algebra, only if

$$4(u+x) \;=\; (u-x)(x-v)(x+v).$$

Now, depending on the values of $u$ and $v$, this equation has no positive solutions for $x$, two positive solutions for $x$ (both of which are between $v$ and $u$), or exactly one positive solution for $x$ (which is between $v$ and $u$). In the first case there is no equable triangle with one side of length $a$ and another side of length $b$, in the second case there are two equable triangles with one side of length $a$ and another side of length $b$, and in the third case there is exactly one equable triangle with one side of length $a$ and another side of length $b$. Thus, we need to classify all values of $u$ and $v$ such that this equation has exactly one positive solution.

We could proceed as in the above solution, but another method would be to write our equation as

$$(x-u)(x-v)(x+v) + 4(x+u) \;=\; 0$$

and set the discriminant of the left-hand side equal to 0. Doing this (I used WolframAlpha), we get

$$(v^2+4)u^4 - 2(v^4+28v^2+88)u^2 + (v^2-4)^3 \;=\; 0.$$

The only positive solution for $u$ with $u > v$ is given by

$$u \;=\; \sqrt{\frac{v^4+28v^2+88+8\sqrt{(v^2+5)^3}}{v^2+4}}.$$

To make this look a bit nicer, put $k = \sqrt{v^2+5}$. Then we have (again with help from WolframAlpha)

$$u \;=\; (k+3)\sqrt{\frac{k+3}{k+1}} \qquad \text{and} \qquad v \;=\; \sqrt{k^2-5}.$$

Putting this all together, we can classify our "unique equable pairs" $(a,b)$ by

$$a \;=\; (k+3)\sqrt{\frac{k+3}{k+1}} + \sqrt{k^2-5}$$

and

$$b \;=\; (k+3)\sqrt{\frac{k+3}{k+1}} - \sqrt{k^2-5},$$

where $k \geq \sqrt{5}$. (The corresponding third side of the triangle can be shown to be $2x = 2\sqrt{(k+3)(k+1)}$.)

**#1351:** *Proposed by Hongwei Chen, Christopher Newport University.*

Let $\zeta$ be the Riemann zeta function; $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3 = 1.2020569\ldots$ is now known as *Apéry constant* since Roger Apéry first proved that $\zeta(3)$ is irrational in 1979. Since then, more efforts have been focused on seeking a rapidly convergent series. Show that

$$\zeta(3) = 1 + \sum_{n=1}^{\infty} \frac{1}{n^3+4n^7}.$$

This series has the convergence rate $O(n^{-7})$ instead of $O(n^{-3})$.

*Solution below by the proposer; first correct solution received by* ***Kenny Davenport****. Also solved by* ***Brian D. Beasley, Presbyterian College****.*

The partial fraction decomposition

$$\frac{1}{x^3(1+4x^4)} \;=\; \frac{1}{x^3} + \frac{1}{2x^2+2x+1} - \frac{1}{2x^2-2x+1},$$

yields

$$\begin{aligned} \frac{1}{n^3(1+4n^4)} \;&=\; \frac{1}{n^3} + \frac{1}{2n^2+2n+1} - \frac{1}{2n^2-2n+1} \\ &=\; \frac{1}{n^3} + \frac{1}{2n^2+2n+1} - \frac{1}{2(n-1)^2+2(n-1)+1}. \end{aligned}$$

Thus, by telescoping, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^3+4n^7} &= \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty}\left(\frac{1}{2n^2+2n+1} - \frac{1}{2(n-1)^2+2(n-1)+1}\right) \\ &= \zeta(3)-1,\end{aligned}$$

which is equivalent to the claimed identity. This completes the proof.

**#1352:** *Proposed by Pete Schumer, Middlebury College.*
The following is from the 2017 Green Chicken Math Competition between Middlebury and Williams Colleges. If the length of the side of a triangle is less than the average lengths of the other two sides, show that the opposite angle is less than the average of the other two angles.
*Solution by* ***Ioannis D. Sfikas, Athens, Greece****. Also solved by* ***Levent Adil Batakci, Hershey High School****,* ***Jennifer Yager, North Central College****.* x Let $a = BC, b = AC$ and $c = AB$ be the lengths of the triangle $ABC$, with opposite angles $A, B$ and $C$. From the given information, we order the sides so that

$$a \ \leq \ \frac{b+c}{2}$$

(call this equation (1)). The law of sines states that

$$\frac{a}{\sin A} \ = \ \frac{b}{\sin B} \ = \ \frac{c}{\sin C} \ = \ 2R,$$

where $R$ is the diameter of the triangle's circumcircle. Applying the law of sines with (1) yields

$$2R\sin A \ \leq \ \frac{2R\sin B + 2R\sin C}{2} \quad \text{or} \quad \sin A \ \leq \ \frac{\sin B + \sin C}{2}.$$

If $p_1, \dots, p_n$ are positive numbers which sum to 1 and $f(x)$ is a real continuous function that is concave, then by Jensen's inequality we have

$$\sum_{p=1}^{n} p_i f(x_i) \ \leq \ f\left(\sum_{p=1}^{n} p_i x_i\right).$$

Since the function $f(x) = \sin x$ is concave in the interval $(0, \pi)$, we find

$$\frac{\sin B + \sin C}{2} \ \leq \ \sin\left(\frac{B+C}{2}\right) \quad \text{and} \quad \sin A \ \leq \ \frac{\sin B + \sin C}{2} \ \leq \ \sin\left(\frac{B+C}{2}\right).$$

Thus, we conclude that $A \leq (B+C)/2$, completing the proof. (Note: The same problem appears in *Index to Mathematical Problems, 1980-1984* by Stanley Rabinowitz, MathPro Press, page 140.)

**#1353:** *Proposed by Kenny Davenport.*

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the $n^{\text{th}}$ Catalan number, and $L_n$ equal the $n^{\text{th}}$ Lucas number (these are given by the recurrence $L_{n+1} = L_n + L_{n-1}$ with initial conditions $L_0 = 2$ and $L_1 = 1$). Find

$$\sum_{n=0}^{\infty} nC_{n+1}L_n/8^n.$$

*Solution by* **Hongwei Chen, Christopher Newport University**. *Also solved by* **Ioannis D. Sfikas, Athens, Greece** *and* **Brian Bradie, Christopher Newport University**.

We determine the sum value by using the generating function of $\{nC_{n+1}\}$. Recall the well-known formula

$$\sum_{n=0}^{\infty} \binom{2n}{n} t^n = \frac{1}{\sqrt{1-4t}}.$$

Integrating this equation from 0 to $x$ gives

$$\sum_{n=0}^{\infty} C_n x^{n+1} = \frac{1-\sqrt{1-4x}}{2}.$$

Dividing both sides by $x^2$ and then moving $1/x$ to the right-hand side yields

$$\sum_{n=0}^{\infty} C_{n+1}x^n = \frac{1-\sqrt{1-4x}}{2x^2} - \frac{1}{x} = \frac{1-2x-\sqrt{1-4x}}{2x^2}.$$

Finally, applying $x\frac{d}{dx}$ obtains the generating function of $\{nC_{n+1}\}$:

$$G(x) := \sum_{n=0}^{\infty} nC_{n+1}x^n = \frac{1-3x-(1-x)\sqrt{1-4x}}{x^2\sqrt{1-4x}}. \tag{1}$$

Let $\phi = (1+\sqrt{5})/2, \bar{\phi} = (1-\sqrt{5})/2$. Then $L_n = \phi^n + \bar{\phi}^n$. By (1), we have

$$\sum_{n=0}^{\infty} nC_{n+1}L_n/8^n = G(\phi/8) + G(\bar{\phi}/8). \tag{2}$$

Since

$$\sqrt{1-4\phi/8} = \sqrt{1-\phi/2} = \frac{1}{2}\sqrt{3-\sqrt{5}} = \frac{\sqrt{2}}{4}(1-\sqrt{5});$$
$$\sqrt{1-4\bar{\phi}/8} = \sqrt{1-\bar{\phi}/2} = \frac{1}{2}\sqrt{3+\sqrt{5}} = \frac{\sqrt{2}}{4}(1+\sqrt{5});$$

this, together with (2), leads to

$$\sum_{n=0}^{\infty} nC_{n+1}L_n/8^n = 64\sqrt{10} - 200.$$

**#1354:** *David Benko, University of South Alabama.*

There is a road around a lake with 9 gas stations along it, whose locations can be arbitrarily fixed. We have a map of the road which also states how much gasoline is at each gas station. We know how many miles per gallon our car gets, and interestingly it turns out that the total amount of gasoline at the stations is exactly enough to go around the lake once. Starting with an empty tank, can we always choose a gas station such that if a helicopter (carefully!) drops our car at that gas station, we can use it as a starting point to go around the lake, arriving back to the initial gas station? We assume that there are no other cars on the road, the gasoline is free, the size of our gasoline tank is unlimited, and we always get the same miles per gallon throughout our trip.

*Solution by* ***Ashland University Undergraduate Problem Solving Group****. Also solved by* ***Timothy O'Neill and David Schmitz at North Central College****.*

Consider the gas stations on a circle, with $X_i$ the location of the $i^{\text{th}}$ gas station with $G_i$ units of gas, measured in units of the number of miles the car can travel using the gas. Let $D_i$ be the distance between $X_i$ and $X_{i+1}$. We are given that

$$D_1 + D_2 + \cdots + D_9 \ = \ G_1 + G2 + \cdots + G_9.$$

Let's start at $X_1$ and move towards $X_2$ ; we continue around the lake until we run out of gas or until we get back to $X_1$. If we don't make it all the way around the island, then we end up with a disjointed graph, where one piece is from $X_1$ to the last gas station we are able to reach before running out of gas and the other being the rest of the graph. For example, let's suppose that we make it to $X_2$ , but don't make it to $X_3$. We have one piece where $D_1 + D_2 > G_1 + G_2$ and one where $D_3 + D_4 + \cdots + D_9 < G_3 + G_4 + \cdots + G_9$. The additional gas needed to make it from $X_2$ to $X_3$ is contained by the gas stations from $X_3$ to $X_9$. We now start at the gas station we couldn't reach, in this case, $X_3$, and move towards $X_4$. Either we make it to $X_1$ from $X_3$ where we now have the extra gas needed to make it to $X_3$ from $X_1$ or we don't make it to one of the gas stations between $X_4$ and $X_1$. If we don't make it to $X_1$ then we have another disjointed piece of the graph. For example, let's suppose we make it from $X_3$ to $X_5$, but don't have enough gas to make it to $X_6$. We have one piece where $D_1 + D_2 > G_1 + G_2$ , a piece where $D_3 + D_4 + D_5 > G_3 + G_4 + G_5$, and a third piece where $D_6 + D_7 + D_8 + D_9 < G_6 + G_7 + G_8 + G_9$. This means the additional gas needed to make it from $X_2$ to $X_3$ when starting at $X_1$ and to make it from $X_5$ to $X_6$ when starting at $X_3$ is contained by the gas stations $X_6$ to $X_9$. We now consider starting at gas station $X_6$ and repeat this approach until we eventually are able to make it to $X_1$ , with all of the extra gas necessary to connect the disjointed pieces of the graph. The gas station at which we start when we finally reach $X_1$ is the one where the helicopter should drop our car.

*Solution by the proposer:* We proceed by induction. Clearly if there is just 1 station it can be done. Suppose we can do $n$ stations placed at any locations. If starting at any station cannot take us to the next one, than the total amount of gasoline is not enough for the trip, clearly. Now place one more station. Clearly there is enough gas to get from at least one station to the next, or we cannot make it around. Say there is enough gas to get from station $i$ to station $i + 1$. Then remove station $i + 1$ and put all of its gas at station $i$. We now have $n$ stations, and by induction there is a solution.

**GRE Practice #4:** *Proposed by Steven Miller, Williams College*

One of the greatest challenges students have with the math GRE subject test is that while they solve a problem, often it is faster to eliminate four wrong answers than find the exact solution (or at least eliminate a few answers, at which point on average it is advantageous to

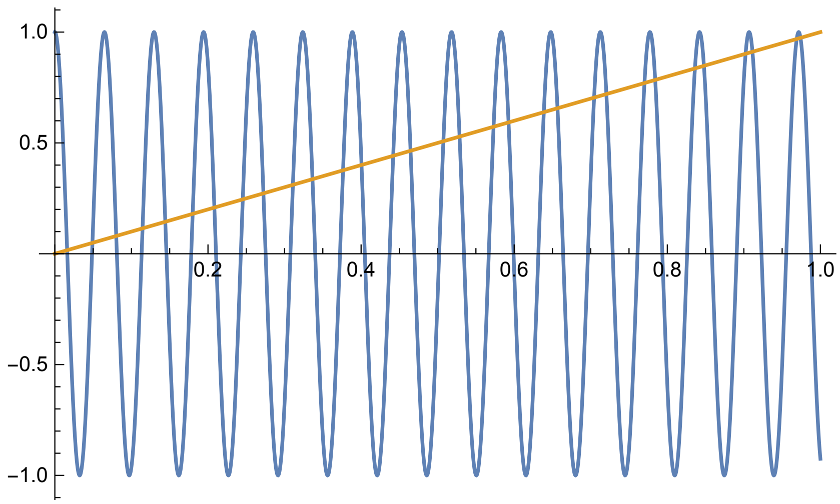


FIGURE 3. Plot of $\cos(97x)$ and $x$ on $[0,1]$.

guess). Consider the following (a discussion of the answer is included after the solutions to earlier PME problems), taken from one of the on-line collections of GRE problems (it was Problem 45). How many positive numbers $x$ satisfy the equation $\cos(97x) = x$? The options are (a) 1, (b) 15, (c) 31, (d) 49, (e) 96.

**Solution:** For this problem it's useful to know that $\pi \approx 3.14$, and thus $2\pi \approx 6.28$ (this is also known at $\tau$, and has advocates who celebrate it on June 28th). Since the absolute value of cosine is at most 1, once $x$ exceeds 1 there cannot be any more solutions. Thus we just need to look for $x \in [0,1]$.

Note that cosine is periodic; how many periods does $\cos(97x)$ go through when $x$ ranges from 0 to 1? Since $2\pi \approx 6.28$, we're looking for how many times 6.28 goes into 97. As a rough estimate the answer is between 97/7 (which is a little less than 14) and 97/6 (which is a tad more than 16). We're thus looking at somewhere between 14 and 17 periods of cosine. As cosine ranges from 1 down to -1 and then back up to 1 in each period, we expect to have two solutions to $\cos(97x) = x$ in each complete period entirely contained in $[0,1]$. Thus there should be around 28 to 34 solutions; maybe a little higher as we could have a partial period at the end giving a solution or two, so we expect the answer to be between 28 and 36. Only one choice is in that range, (c) 31. We plot the two functions in Figure 3, and see the answer is indeed 31.

*E-mail address*: sjm1@williams.edu

PROFESSOR OF MATHEMATICS, DEPARTMENT OF MATHEMATICS AND STATISTICS, WILLIAMS COLLEGE, WILLIAMSTOWN, MA 01267