PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2021

STEVEN J. MILLER (EDITOR)

1. Problems: Fall 2021

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). No school submitted two correct answers this time around, but hopefully as pandemic response restrictions are relaxed more schools will have more in-person meetings, and more submissions.... Also in the last issue one problem solver was accidentally omitted; Problem #1359 was also solved by Kenneth Davenport (SCI-Dallas, Dallas, PA).

Each year a distinguished mathematician gives the J. Sutherland Frame Pi Mu Epsilon Lecture at MathFest. In 2019 that speaker was Alice Silverberg, Distinguished Professor at the University of California, Irvine, and Problem #1366 is inspired by her lecture. For 2020 the speaker was supposed to be Florian Luca from the University of the Witwatersrand. His talk was rescheduled due to the pandemic response to Mathfest 2021. The abstract for his talk, *Arithmetic and Digits*, is the following: *In our recent paper in the Monthly (October,*

Date: October 26, 2021.

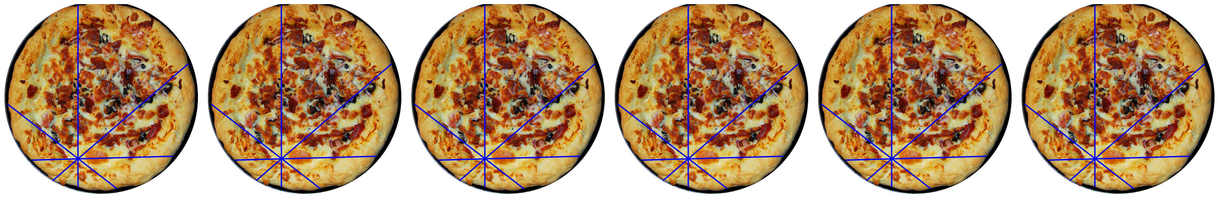

Figure 1. Pizza motivation; can you name the theorem that's represented here?

2019) with Pante Stănică, we looked at perfect squares which arise when concatenating two consecutive positive integers like $183184 = 428^2$ *with the smaller number to the left, or* $98029801 = 9901^2$ *with the larger number to the left. My talk will present variations on this topic with the aim of providing the audience with examples of numbers which are both arithmetically interesting (like perfect squares) while their digital representations obey some regular patterns. The examples will not be limited to perfect squares, but will also include other old friends like Fibonacci numbers and palindromes.* His talk is available online at , and related to that is the following bonus problem:

> Let $f(x) = ax^2 + bx + c$ be a polynomial with integer coefficients and positive leading term. Find conditions on a, b, c such that $f(x) = \overline{x(x+1)}$ has infinitely many positive integer solutions x. Here $\overline{x(x+1)}$ is the concatenation of x with $x+1$ as in our paper. (Thus $\overline{20(21)} = 2021$.)

One final note: I would like to express my thanks to George Stoica for submitting all the problems for this issue.

#1377: *Proposed by George Stoica, Saint John, New Brunswick, Canada.* Consider a decreasing and positive sequence $\{a_{2k-1}\}_{k\geq 1}$. Prove there exists another decreasing and positive sequence $\{a_{2k}\}_{k\geq 1}$ such that the combined sequence $\{a_n\}_{n\geq 1}$ is decreasing and the series $\sum_{n=1}^{\infty} a_n$ converges to an irrational number.

#1378: *Proposed by George Stoica, Saint John, New Brunswick, Canada.* Let $A \in \mathcal{M}_n(\mathbb{C})$, the set of $n \times n$ complex matrices, be such that $\det(A^k + I_n) = 1$ for any positive integer k. Prove that $A^n = O_n$, the $n \times n$ zero matrix.

#1379: *Proposed by George Stoica, Saint John, New Brunswick, Canada.* Let $(x_n)_n$ be a sequence of real numbers, uniformly distributed modulo 1 and non-decreasing. Prove that the sequence $(x_n/\log n)_{n\geq 2}$ is unbounded. Can one replace $(\log n)_{n\geq 2}$ by another sequence that approaches ∞ faster than $\log n$?

#1380: *Proposed by George Stoica, Saint John, New Brunswick, Canada.* It is very well known that, if $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ exists, then $\displaystyle\lim_{n\to\infty} a_n = 0$. Prove that the conclusion may be false under the (weaker) hypothesis that $\lim_{n\to\infty} \sum_{i=[n/2]+1}^{n} a_i$ exists. (Square brackets denote the integer part).

#1381: *Proposed by George Stoica, Saint John, New Brunswick, Canada.* Is it true that $x_n \to 0$ if $x_{n+1} = |x_n - a_n|$ for all $n \geq 1$, with $a_n \searrow 0$ and $\sum_{n=1}^{\infty} a_n = \infty$?

GRE Practice #8 Consider the polynomial

$$f(x) \;=\; 6x^8 - 7x^7 - 25x^6 + 30x^5 + 16x^4 + 8x^3 - 37x^2 - x + 10.$$

As the coefficients of f are relatively prime, by Gauss' Lemma if we can write $f(x)$ as a product of two polynomials with rational coefficients then we can write f as a product of two polynomials with integer coefficients. Which polynomial below divides $f(x)$?
(a) $x^5+4x^4-2x^2+3$ (b) x^4-3x^3+2x-4 (c) $2x^5-4x^3-3x^2+4$ (d) x^5-4x^3+2x+5
(e) $2x^3-4x^2+x+6$.

2. Solutions

#1371: *Proposed by Harold Reiter (University of North Carolina Charlotte).*
A set of chess knights is called *independent* if none of them attack any of the others.

(1) How many ways can 6 independent knights be placed on an 4×4 chess board?
(2) How many ways can 8 independent knights be placed on an 4×4 chess board?
(3) How many ways can 7 independent knights be placed on an 4×4 chess board?
(4) Prove that nine knights cannot be arranged independently on a 4×4 chess board.

Solved by ***Watson Houck, Barringer Academic Center.***
As the solution is long and detailed, it is posted here: https://web.williams.edu/Mathematics/sjmiller/public_html/pme/PMEknights_sept4th.pdf.

#1374: *Proposed by Himadri Lal Das, Indian Institute of Technology Kharagpur, India.*
Let $f_n : [0,1) \to \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$, be a family of functions, where $f_n(x)$ is the number of nonzero terms up to n decimal places of $x \in [0,1)$. Let

$$c = 0.1010010001\ldots$$

find a closed form expression for $f_n(c)$ and calculate $\lim_{n\to\infty} \frac{f_n(c)}{\sqrt{n}}$. Here $\mathbb{N}$ is the collection of all positive integers.
Solved by ***Henry Ricardo, Westchester Area Math Circle.*** *Also solved by* ***Brian D. Beasley, Presbyterian College.***

We observe that $c = \sum_{k=1}^{\infty} 10^{-k(k+1)/2}$—that is, the digit 1 appears at the decimal places corresponding to triangular numbers. Therefore $f_n(c)$ counts the number of triangular numbers $k(k+1)/2$, $k\in\mathbb{N}$, that are less than or equal to n:

$$\frac{k(k+1)}{2} \;\leq\; n \iff k^2+k-2n \;\leq\; 0 \;\text{ if and only if }\; k \;\leq\; \frac{-1+\sqrt{1+8n}}{2}.$$

Since k must be an integer, we have

$$f_n(c) \;=\; \left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Thus we have

$$\frac{-1+\sqrt{1+8n}-2}{2\sqrt{n}} \;<\; \frac{f_n(c)}{\sqrt{n}} \;\leq\; \frac{-1+\sqrt{1+8n}}{2\sqrt{n}}.$$

Since

$$\lim_{n\to\infty} \frac{-1+\sqrt{1+8n}-2}{2\sqrt{n}} \;=\; \lim_{n\to\infty} \frac{-1+\sqrt{1+8n}}{2\sqrt{n}} \;=\; \frac{1}{2}\sqrt{8} \;=\; \sqrt{2},$$

the Squeeze Theorem yields $f_n(c)/\sqrt{n} \to \sqrt{2}$ as $n \to \infty$.

#1376: *Proposed by Hongwei Chen, Christopher Newport University.*
Prove

$$\int_0^1 \int_0^1 \frac{x\ln(x)\ln(y)}{(1+x^2)(1+xy)}\,dxdy \;=\; \frac{1}{48}\pi^2 G,$$

where $G = \sum_{k=0}^{\infty} (-1)^k/(2k+1)^2$ is the Catalan constant. This problem is originated from the study of generalization of Euler sums, which has recently been an active research topic. For example, we have

$$\int_0^1 \frac{\ln x \mathrm{Li}_2(-x)}{1+x^2}\,dx \;=\; \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+n+1)^2},$$

where $\mathrm{Li}_2(x)$ is the Dilogarithm function defined by

$$\mathrm{Li}_2(z) \;=\; \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

Since

$$\mathrm{Li}_2(x) \;=\; -\int_0^1 \frac{x\ln y}{1-xy}\,dy,$$

we have

$$\int_0^1 \frac{\ln x \mathrm{Li}_2(-x)}{1+x^2}\,dx \;=\; \int_0^1 \int_0^1 \frac{x\ln(x)\ln(y)}{(1+x^2)(1+xy)}\,dxdy.$$

Solved by ***Seán M. Stewart, Bomaderry, NSW, Australia***.
Denote the integral whose value is to be proved by I. We begin by writing the integral as follows

$$I \;=\; \frac{1}{2}\int_0^1 \int_0^1 \frac{x\log(x)\log(y)}{(1+x^2)(1+xy)}\,dx\,dy + \frac{1}{2}\int_0^1 \int_0^1 \frac{x\log(x)\log(y)}{(1+x^2)(1+xy)}\,dx\,dy.$$

By symmetry found in the integrand an interchange between the two dummy variables appearing in the right most double integral yields

$$I \;=\; \frac{1}{2}\int_0^1 \int_0^1 \frac{x\log(x)\log(y)}{(1+x^2)(1+xy)}\,dx\,dy + \frac{1}{2}\int_0^1 \int_0^1 \frac{y\log(y)\log(x)}{(1+y^2)(1+yx)}\,dy\,dx,$$

or

$$I = \frac{1}{2}\int_0^1\int_0^1 \frac{x\log(x)\log(y)}{(1+x^2)(1+xy)}\,dx\,dy + \frac{1}{2}\int_0^1\int_0^1 \frac{y\log(x)\log(y)}{(1+y^2)(1+xy)}\,dx\,dy,$$

after the order of integration in the right most integral has been changed. Thus

$$\begin{aligned}
I &= \frac{1}{2}\int_0^1\int_0^1 \left(\frac{x}{1+x^2}+\frac{y}{1+y^2}\right)\frac{\log(x)\log(y)}{1+xy}\,dx\,dy\\
&= \frac{1}{2}\int_0^1\int_0^1 \frac{(x+y)(1+xy)\log(x)\log(y)}{(1+x^2)(1+y^2)(1+xy)}\,dx\,dy\\
&= \frac{1}{2}\int_0^1\int_0^1 \frac{(x+y)\log(x)\log(y)}{(1+x^2)(1+y^2)}\,dx\,dy\\
&= \frac{1}{2}\int_0^1\int_0^1 \frac{x\log(x)\log(y)}{(1+x^2)(1+y^2)}\,dx\,dy + \frac{1}{2}\int_0^1\int_0^1 \frac{y\log(x)\log(y)}{(1+x^2)(1+y^2)}\,dx\,dy\\
&= \frac{1}{2}\int_0^1\int_0^1 \frac{x\log(x)\log(y)}{(1+x^2)(1+y^2)}\,dx\,dy + \frac{1}{2}\int_0^1\int_0^1 \frac{x\log(y)\log(x)}{(1+y^2)(1+x^2)}\,dy\,dx,
\end{aligned}$$

where in the last line we have exploited the symmetry found in the integrand. Changing the order of integration in the right most integral yields

$$I = \int_0^1\int_0^1 \frac{x\log(x)\log(y)}{(1+x^2)(1+y^2)}\,dx\,dy = \left(\int_0^1 \frac{x\log(x)}{1+x^2}\,dx\right)\cdot\left(\int_0^1 \frac{\log(y)}{1+y^2}\,dy\right). \tag{2.1}$$

Each of the integrals that have appeared can be readily found. For the first of these

$$\int_0^1 \frac{x\log(x)}{1+x^2}\,dx = \int_0^1 \sum_{k=0}^\infty (-1)^k x^{2k+1}\log(x)\,dx = \sum_{k=0}^\infty (-1)^k \int_0^1 x^{2k+1}\log(x)\,dx.$$

Here the interchange made between the summation and integration can be justified using the dominated convergence theorem. Integrating by parts twice we find

$$\int_0^1 \frac{x\log(x)}{1+x^2}\,dx = -\frac{1}{4}\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2} = \frac{1}{4}\sum_{k=1}^\infty \frac{(-1)^k}{k^2},$$

where a reindexing of the sum of $k \mapsto k-1$ has been made. As

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2} + \sum_{k=1}^\infty \frac{1}{k^2} = \sum_{k=1}^\infty \frac{(-1)^k+1}{k^2} = \sum_{k=1}^\infty \frac{2}{(2k)^2},$$

we see that

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^2} = \frac{1}{2}\sum_{k=1}^\infty \frac{1}{k^2} - \sum_{k=1}^\infty \frac{1}{k^2} = -\frac{1}{2}\sum_{k=1}^\infty \frac{1}{k^2} = -\frac{\pi^2}{12},$$

giving

$$\int_0^1 \frac{x\log(x)}{1+x^2}\,dx = -\frac{\pi^2}{12}\cdot\frac{1}{4} = -\frac{\pi^2}{48}.$$

A similar thing can be done for the second of the integrals. Here we have

$$\int_0^1 \frac{\log(y)}{1+y^2}\,dy = \int_0^1 \sum_{k=0}^\infty (-1)^k y^{2k}\log(y)\,dy = \sum_{k=0}^\infty (-1)^k \int_0^1 y^{2k}\log(y)\,dy.$$

Here the interchange made between the summation and integration can again be justified using the dominated convergence theorem. Integrating by parts twice yields

$$\int_0^1 \frac{\log(y)}{1+y^2}\,dy \;=\; -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \;=\; -\mathbf{G}.$$

Combining the results found for the two integrals into (2.1) yields

$$I \;=\; \left(-\frac{\pi^2}{48}\right)\cdot(-\mathbf{G}) \;=\; \frac{1}{48}\pi^2\mathbf{G},$$

as required to prove.

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As the coefficients of f are relatively prime, by Gauss' Lemma if we can write $f(x)$ as a product of two polynomials with rational coefficients then we can write f as a product of two polynomials with integer coefficients. Which polynomial below divides $f(x)$?
(a) $x^5+4x^4-2x^2+3$ (b) x^4-3x^3+2x-4 (c) $2x^5-4x^3-3x^2+4$ (d) x^5-4x^3+2x+5 (e) $2x^3-4x^2+x+6$.

We could of course factor this. If we use the rational root test, we find that if p/q is a root then $q|6$ (the leading term) and $p|10$ (the constant term); if you remember something like the test but not the exact statement you can often "find" it by looking at a special case; here if we look at $3x-2=0$ we find the root is $p/q = 2/3$, and thus $p|2$ while $q|3$. Doing this, we find the possibilities for rational roots are $\pm1, \pm2, \pm5, \pm10$, $\pm1/2$, $\pm5/2$, $\pm1/3$, $\pm2/3$, $\pm5/3$, $\pm10/3$, $\pm1/6$, $\pm5/6$. This is a long list! After work we find $-1/2, 1, 2/3$ are roots, and thus $f(x)$ is divisible by $(2x+1)(x-1)(3x-2)$. Doing the long division, we see

$$f(x) \;=\; (2x+1)(x-1)(3x-2)\cdot(x^5-4x^3+2x+5),$$

and thus the answer is (d).

Would it be faster to just try the five possibilities and see which one works? Probably! In this case I would probably try (d) or (e) first, under the assumption that it is unlikely they would have (a) work as that would cut down on the time.

Of course, as the point of these problems is to show faster ways to find the answer, there is a better approach. We do *not* need to factor $f(x)$; we need to find which of the five candidates is a factor. We are thus told one and only one works. All we need to do is find a way to eliminate four of the five options. The simplest way is to specialize x. If we take $x=0$ we get $f(x)=10$, so whatever of the five choices is a factor, its specialization must divide. The five options, at $x=0$, are 3, -4, 4, 5 and 6; of these only 5 divides 10, and thus the answer is (d)!

We might have been unlucky and had two or more candidates successfully divide when $x=0$. If that had happened, try specializing to $x=\pm1$. If we still have more than one option in play, try ±2....

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