# PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2021

STEVEN J. MILLER (EDITOR)

## 1. Problems: Spring 2021

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). Due to the pandemic, many campus problem solving groups have not met; hopefully more solutions will come from these problems! (The Fall '20 problems were repeated from Spring '20.)

Each year a distinguished mathematician gives the J. Sutherland Frame Pi Mu Epsilon Lecture at MathFest. In 2019 that speaker was Alice Silverberg, Distinguished Professor at the University of California, Irvine. For 2020 the speaker was to be Florian Luca from the University of the Witwatersrand and we were to have a problem based on his talk, but MathFest canceled due to the covid response; we hope to share such inspired problems again in the future.

*Date*: March 18, 2021.

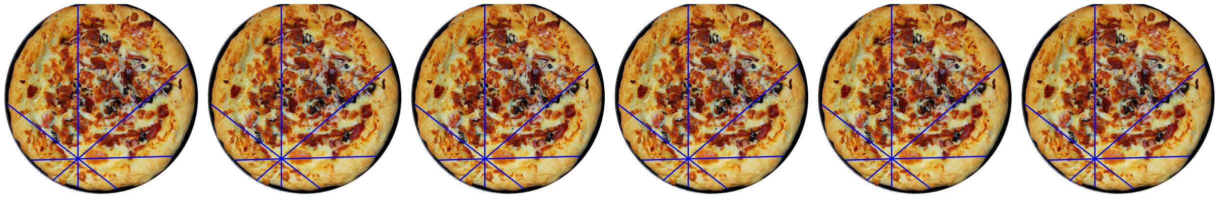

FIGURE 1. Pizza motivation; can you name the theorem that's represented here?

**#1371:** *Proposed by Harold Reiter (University of North Carolina Charlotte).*

A set of chess knights is called *independent* if none of them attack any of the others.

(1) How many ways can 6 independent knights be placed on an $4 \times 4$ chess board?
(2) How many ways can 8 independent knights be placed on an $4 \times 4$ chess board?
(3) How many ways can 7 independent knights be placed on an $4 \times 4$ chess board?
(4) Prove that nine knights cannot be arranged independently on a $4 \times 4$ chess board.

**#1372:** *Proposed by Steven J. Miller and Chenyang Sun (Williams College).*
The following is a standard problem, with a generalization (see #1373) that has applications in understanding some methods in Operations Research. Consider a positive integer $N \geq 100$. (a) We want to divide $N$ into positive integer pieces $a_1, \dots, a_n$ such that the product $a_1 \cdots a_n$ is as large as possible. How do we do this? (b) What if we just require the pieces to be positive numbers: how do we do it in that case?

**#1373:** *Proposed by Steven J. Miller and Chenyang Sun (Williams College).*
In the previous problem, one solution involves finding the location of integer with largest output for a function. Often this integer is the the integer either immediately before or after the real maximum. Consider the following two dimensional problem. Maximize the function $f(x,y) = \frac{200}{(5x+2)(5y+2)}$ over the positive integers, subject to the constraint $xy \geq 5$. If we just wanted its maximum (so $xy \geq 5$ but $x$ and $y$ need not be integers), where would that be?

**#1374:** *Proposed by Himadri Lal Das, Indian Institute of Technology Kharagpur, India.*
Let $f_n : [0,1) \to \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, be a family of functions, where $f_n(x)$ is the number of nonzero terms up to $n$ decimal places of $x \in [0,1)$. Let

$$c = 0.1010010001\ldots$$

find a closed form expression for $f_n(c)$ and calculate $\lim_{n\to\infty} \frac{f_n(c)}{\sqrt{n}}$. Here $\mathbb{N}$ is the collection of all positive integers.

**#1375:** *Proposed by Charles Audet, Ecole Polytechnique de Montréal.*

A collection of $n$ squares are placed side-by-side. They occupy an area greater than or equal to 1, and the sum of their side lengths does not exceed an integer $k \leq n$. Show that there are $k$ squares whose sum of side lenghts is greater than or equal to 1.

**#1376:** *Proposed by Hongwei Chen, Christopher Newport University.*

Prove

$$\int_0^1 \int_0^1 \frac{x\ln(x)\ln(y)}{(1+x^2)(1+xy)}\,dxdy \;=\; \frac{1}{48}\pi^2 G,$$

where $G = \sum_{k=0}^{\infty}(-1)^k/(2k+1)^2$ is the Catalan constant. This problem is originated from the study of generalization of Euler sums, which has recently been an active research topic. For example, we have

$$\int_0^1 \frac{\ln x \mathrm{Li}_2(-x)}{1+x^2}\,dx \;=\; \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+n+1)^2},$$

where $\mathrm{Li}_2(x)$ is the Dilogarithm function defined by

$$\mathrm{Li}_2(z) \;=\; \sum_{k=1}^{\infty}\frac{x^k}{k^2}.$$

Since

$$\mathrm{Li}_2(x) \;=\; -\int_0^1 \frac{x\ln y}{1-xy}\,dy,$$

we have

$$\int_0^1 \frac{\ln x \mathrm{Li}_2(-x)}{1+x^2}\,dx \;=\; \int_0^1\int_0^1 \frac{x\ln(x)\ln(y)}{(1+x^2)(1+xy)}\,dxdy.$$

**GRE Practice #7:** The following is taken from Practice Problem #3 at https://www.ets.org/s/gre/pdf/practice_book_math.pdf. Find

$$\int_{e^{-3}}^{e^{-2}} \frac{1}{x\log x}dx.$$

(a) 1 (b) 2/3 (c) 3/2 (d) log(2/3) (e) log(3/2).

## 2. Solutions

**GRE Practice #7:** The following is taken from Practice Problem #3 at https://www.ets.org/s/gre/pdf/practice_book_math.pdf. Find

$$\int_{e^{-3}}^{e^{-2}} \frac{1}{x\log x}dx.$$

(a) 1 (b) 2/3 (c) 3/2 (d) log(2/3) (e) log(3/2).

One could find the anti-derivative of $1/(x\log x)$ and do the integration. If you notice it is $(\log x)^{-1}d\log x/dx$ then the anti-derivative is $\log\log x + c$ for any constant $c$. We may take $c = 0$, and the answer is thus $\log\log(e^{-3}) - \log\log(e^{-2})$. If we try to naively evaluate this as a real valued function we run into trouble, as $\log\log(e^{-3}) = \log(-3)$, and we cannot take the logarithm of a negative number! We can salvage this with passing to the complex plane, but you are hopefully seeing that this is not the best way to go.

How else could we attack the problem? Whenever you have an integral, try to get a sense of its size or value. Note that the integrand is *negative* in the entire range of integration, thus the answer must be negative! For example, if $f(x) = 1/(x\log x)$ then $f(e^{-3}) = -e^3/3$, and similarly $f(e^{-2}) = -e^2/2$. Notice the only answer in the five which is negative is (d) $\log(2/3)$.

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