# PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2022 

STEVEN J. MILLER (EDITOR)

## 1. Problems: Spring 2022

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk $\left({ }^{*}\right)$ preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two poblems from the current issue will be entered in a lottery to win a pizza party (value up to $\$ 100$ ). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). The winning school is Case Western Reserve University.


Figure 1. Pizza motivation; can you name the theorem that's represented here?

Each year a distinguished mathematician gives the J. Sutherland Frame Pi Mu Epsilon Lecture at MathFest. In 2019 that speaker was Alice Silverberg, Distinguished Professor at the University of California, Irvine, and Problem \#1366 is inspired by her lecture. For 2020 the speaker was to be Florian Luca from the University of the Witwatersrand and we were to have a problem based on his talk, but MathFest canceled due to the covid response and he gave it in 2021, leading to a bonus problem in the Fall 2021 issue. We hope to share such inspired problems again in the Fall 2022 set.
\#1382: Proposed by George Stoica, Saint John, New Brunswick, Canada. Find all integers $a, b, c, d$ such that $a^{3}+b^{3}+c^{3}+d^{3}=2021$; note the integers may be zero or negative. Due to a backlog this problem is appearing in 2022 instead of 2021; if you wish, consider instead $1^{3}+a^{3}+b^{3}+c^{3}+d^{3}=2022$.
\#1383: Proposed by Kenneth Davenport. Consider the following collection of numbers:

$$
\text { (1), } \quad\left(\begin{array}{ll}
1 & 3 \\
5 & 9
\end{array}\right), \quad\left(\begin{array}{ccc}
1 & 3 & 7 \\
5 & 9 & 15 \\
11 & 17 & 25
\end{array}\right), \ldots
$$

that is to say in the $n \times n$ matrix we have $a_{i j}$, the entry in the $i^{\text {th }}$ row and $j^{\text {th }}$ column, equals $j^{2}+(2 i-3) j+\left(i^{2}-i+1\right)$.
(a) For what values of $n$ is the sum of the elements of the $n \times n$ matrix a square? For example, if $n=25$ the sum is $18255=135^{2}$.
(b) Consider the sum of the elements on the non-main diagonal:

$$
\begin{aligned}
1 & =1^{3} \\
3+5 & =2^{3} \\
7+9+11 & =3^{3} \\
13+15+17+19 & =4^{3} .
\end{aligned}
$$

Does this pattern continue? If yes prove it; if not, why not (finding a counter-example suffices).
\#1384: Proposed by Steven J. Miller (Williams College). Dirichlet's Theorem of Primes in Arithmetic Progression states that if $a$ and $b$ are relatively prime, then there are infinitely many primes congruent to $b$ modulo $a$ (equivalently, there are infinitely many $n$ such that $a n+b$ is prime). Prove that if this is all we know about primes, it cannot be enough to prove the Twin Prime Conjecture (there are infinitely many primes $p$ such that $p+2$ is also prime). One way to do this is to construct a sequence of integers $\left\{q_{i}\right\}_{i=1}^{\infty}$ with $q_{i+1}>q_{i}+2$ such that for any relatively prime $a$ and $b$ (other than $a=2$ and $b=1$ ) we have infinitely many $n$ such that $a n+b=q_{i}$ for some $i$.
\#1385: Proposed by Hongwei Chen, Christopher Newport University, Newport News, Virginia. A continued fraction is an expression of the form

$$
a_{0}+\frac{1}{a_{1}+\frac{1}{a_{2}+\ldots}}
$$

where the $a_{i}$ are positive integers for all $i \in \mathbb{N}$. It is often denoted by $\left[a_{0} ; a_{1}, a_{2}, \ldots\right]$.
Continued fractions arise naturally in long division and in the theory of approximation to real numbers by rationals. It is known that every finite continued fraction represents a rational number and every periodic continued fraction represents an irrational root of a quadratic equation with integral coefficients. A famous example is $[1,1,1, \ldots]=[\overline{1}]=\phi$, the golden ratio.

This problem is inspired by the recent CMJ Problem 1186 (51:5, 386 (2020)):
Find a closed-form expression for the continued fraction $[1,1, \ldots, 1,3,1,1, \ldots, 1]$, which has $n$ ones before and after, the middle three.

We found the required closed-form expression is

$$
x_{n}:=\frac{F_{n+4} F_{n+1}}{F_{n+2}^{2}}
$$

where $F_{n}$ is the $n$th Fibonacci number.
\#1386: Proposed by Gerard Dion, in memory of Zachary Dion. An interesting construction of Euclidean Geometry is the tangent to two circles with a straightedge and compass. There are 4 possible tangents that are not redundant: 2 for the non-overlapping case (internal and external tangents), 1 for the overlapping case, and 1 for the touching circles case. In the standard construction for non-overlapping circles and external tangent, a new circle is constructed that is inside the larger one, having a radius that is the difference of the other two; see Figure 2.


Figure 2. Construction of tangent to two circles.

Prove the constructions for both the overlapping and non-overlapping cases, presented in Figure 3, also work. One novelty of these is that they remain valid when the construction is done in a tool like Geogebra and the circles are resized to switch which circle is largest.


Figure 3. Construction of tangent to two circles.
\#1387: Proposed by Hongbiao Zeng, Fort Hays State University, Hays, KS. Show that for any positive integer $n$, if $p$ is a prime number such that $n<p<2 n$, then $\binom{2 n}{p} \equiv 1(\bmod p)$.

GRE Practice \#9: The following is Problem \#14 from https://www.ets.org/s/gre/ pdf/practice_book_math.pdf: Suppose $g$ is a continuous real-valued function such that

$$
3 x^{5}+96=\int_{c}^{x} g(t) d t
$$

for each $x \in \mathbb{R}$, where $c$ is a constant. What is the value of $c$ ?
(a) -96
(b) -2
(c) 4
(d) 15
(e) 32 .

## 2. Solutions

\#1370 (originally Spring 2020): Proposed by Eugen J. Ionascu (Columbus State University).
Consider three points $A, B$ and $C$ chosen uniformly at random inside of the region (see Figure 1 below, the yellow region)

$$
\begin{gathered}
\mathcal{R}_{r, \alpha}=\left\{(x, y) \in \mathbb{R}^{2}|x=t \cos \theta, y=t \sin \theta, \pi \geq|\theta| \geq \alpha, t \in[0, r]\},\right. \\
\alpha=\pi a, \quad a \in\left[0, \frac{1}{2}\right], \quad r>0
\end{gathered}
$$

(thus we are choosing from the uniform distribution with respect to the area). Show that the probability that the resulting triangle $\triangle A B C$ contains the origin $O(0,0)$ is equal to

$$
\mathcal{P}_{a}=\frac{(1+a)(1-2 a)^{2}}{4(1-a)^{3}}
$$



## Solved by Volkhard Schindler.

This problem generalizes problem 2048 from Mathematics Magazine, June 2018, p. 230 [Ssorel1] with solution in June 2019, p. 233-234 [Sorel2]. The study of [Sorel2] shows, that the applied ideas are useful to prove problem \#1370, too, by generalization. Obviously, the cited problem concerns the case, that $\left(\alpha \rightarrow \pi / 4=45^{\circ}\right)$, resp. $(a \rightarrow 1 / 4)$, is introduced into the formula of problem $\# 1370$. The probability found in [Sorel2] is $5 / 27$.

A small deviation in the notation of the terms of the problem statement is helpful:
Instead of $\alpha=\pi a$ the angle $\delta=\pi a$ is introduced to describe the half of the angle of the cut-away sector of a unit circle. The angles $\alpha, \beta, \gamma$, are applied to describe the positions of the vertices $A, B, C$, respectively. So, we have to rewrite the region described in the problem statement by:
$\mathcal{R}_{r, \delta}=\left\{(x, y) \in \mathbb{R}^{2}|x=t \cos \theta, y=t \sin \theta, \pi \geq|\theta| \geq \delta, t \in[0, r]\}, \delta=\pi a, a \in\left[0, \frac{1}{2}\right], r=1\right.$.
The probability of the event $\mathcal{E}$, that $O$ lies inside $\triangle A B C$, is equal to $\mathcal{P}_{a}$, i. e. $\mathbf{P}[\mathcal{E}]=\mathcal{P}_{a}$. Let points $P=(+\cos \delta,+\sin \delta)$, and $R=(+\cos \delta,-\sin \delta)$, and angles $\alpha=\angle P O A$, and $\beta=\angle P O B$, and $\gamma=\angle P O C$, so that $\alpha, \beta, \gamma \in[0,2 \pi-2 \delta]$.
Let $\overline{M X}$ be the diameter through $A$ so $A M<A X$ and $\overline{N Y}$ the diameter through $C$ so $C N<C Y$.
Consider the event $\mathcal{E}^{\prime}=\mathcal{E} \cap\{\alpha<\beta<\gamma\}$ as shown in the figure below:
By the assumption that $O$ lies inside $\triangle A B C$, each of the three angles $\angle C O A, \angle A O B$, and $\angle B O C$ must be strictly less than a half revolution. It follows that point $A$ must lie in a sector with angle $(\alpha=\pi-2 \delta), C$ in sector $R O X$ and $B$ in sector $X O Y$, that is:

$$
0<\alpha<\pi-2 \delta, \quad \alpha+\pi<\gamma<2 \pi-2 \delta, \quad \gamma-\pi<\beta<\alpha+\pi .
$$

Observe that the angles $\alpha, \beta, \gamma$, are independent and uniformly distributed in $[0,2 \pi-2 \delta]$ because $A, B, C$ are independent and uniformly distributed in $\mathcal{R}_{r, \delta}$; therefore, the probability

of event $\mathcal{E}^{\prime}$ is:
$\mathbf{P}\left[\mathcal{E}^{\prime}\right]=\int_{0}^{\pi-2 \delta} \int_{\alpha+\pi}^{2 \pi-2 \delta} \int_{\gamma-\pi}^{\alpha+\pi}\left(\frac{1}{2 \pi-2 \delta}\right)^{3} d \beta d \gamma d \alpha=\frac{(\pi+\delta) \cdot(\pi-2 \delta)^{2}}{24(\pi-\delta)^{3}}=\frac{(1+a) \cdot(1-2 a)^{2}}{24(1-a)^{3}}$.
By independence of the three random points $A, B, C$ and the fact that event $\mathcal{E}^{\prime}$ is invariant under permutations thereof, we have obtained what we should have:

$$
\mathbf{P}[\mathcal{E}]=\frac{\mathbf{P}\left[\mathcal{E}^{\prime}\right]}{\mathbf{P}[\{\alpha<\beta<\gamma\}]}=\frac{(1+a) \cdot(1-2 a)^{2}}{24(1-a)^{3}} \cdot \frac{1}{1 / 6}=\frac{(1+a) \cdot(1-2 a)^{2}}{4(1-a)^{3}}
$$

## References:

Sorel1 Julien Sorel: "Proposal of Problem 2048 of the 'Problems' Section"", Mathematics Magazine 91 (2018) 230 (No. 3: June).
Sorel2 Julien Sorel and Xueshi Gao: "Solution of Problem 2048 of the 'Problems' Section":
A random triangle with vertices in a three-quarter disk",
Mathematics Magazine 92 (2019) 233-234 (No. 3: June).
\#1371 (originally Spring 2020): Proposed by Harold Reiter (University of North Carolina Charlotte).

A set of chess knights is called independent if none of them attack any of the others.
(1) How many ways can 6 independent knights be placed on an $4 \times 4$ chess board?
(2) How many ways can 8 independent knights be placed on an $4 \times 4$ chess board?
(3) How many ways can 7 independent knights be placed on an $4 \times 4$ chess board?
(4) Prove that nine knights cannot be arranged independently on a $4 \times 4$ chess board.

Note: The original solution was long and was posted online; we include another approach recently received.

## Solved by Ashland University Problem Solving Group.

We divide our analysis into five cases based on how many knights appear in the middle four squares of the $4 \times 4$ table. This avoids counting duplicate arrangements of knights,
and accounts for every possible arrangement. In the diagrams that follow we use Kn for placed independent knights, $*$ for an intentionally empty square, X for a square that can be attacked by a placed knight, and A, A', for example, for squares that can attack each other.

We consider five cases: four, three, two, one, or zero, respectively, knights in the middle four squares. In each case we calculate the maximum number of independent knights that can be present on the entire board. To clarify, this means the middle four squares will not not be changed during the calculations for each case. The reason the middle four squares are chosen is because any knights placed in the middle of the board are always independent of each other, so any allocation is permissible. For each case we will answer parts 1-3 of the problem, which will finally lead to an answer for part 4.

Case 1: 4 knights in the middle squares.

| X | X | X | X |
| :---: | :---: | :---: | :---: |
| X | Kn | Kn | X |
| X | Kn | Kn | X |
| X | X | X | X |

In this case, every outer square is attacked. So this configuration can have at most 4 independent knights.

Case 2: 3 knights in the middle squares, such as

| X |  | X | X |
| :---: | :---: | :---: | :---: |
|  | Kn | Kn | X |
| X | Kn | $*$ | X |
| X | X | X | X |

Since the two empty squares are not attacked and don't attack each other, this case allows for at most 5 independent knights. For each of the other ways of placing three knights in the middle squares the same will be true.

Case 3: 2 knights in the middle squares. Here there are two distinct configurations: the knights adjacent to each other or diagonal to each other.

Case 3.1. For example

| X | A | B | X |
| :---: | :---: | :---: | :---: |
| $\mathrm{B}^{\prime}$ | Kn | Kn | $\mathrm{A}^{\prime}$ |
| X | $*$ | $*$ | X |
| X | X | X | X |

Independent knights can be placed at either A or $\mathrm{A}^{\prime}$, but not both. Likewise with B and $\mathrm{B}^{\prime}$. Thus for this configuration, and the other three that have knights in adjacent center squares, there are at most of 4 independent knights.

Case 3.2. For example

|  | X |  | X |
| :---: | :---: | :---: | :---: |
| X | Kn | $*$ |  |
|  | $*$ | Kn | X |
| X |  | X |  |

Each of the six non-attacked outer squares is not attacked by any of the others, so we can place up to 8 independent knights in this configuration. The same is true of the configuration which has the knights placed in the other diagonal center squares instead. Thus, there are two ways to get 8 independent knights. Any one of the six non-center knights can be removed to create a configuration with 7 independent knights. Thus, there are $6 \times 2=12$ ways to get 7 independent knights in this case.

There are ${ }_{6} C_{2}=15$ choices to remove two of the non-center knights. Thus there are $15 \times 2=30$ ways to get 6 independent knights in this case.

- 8 knights - in 2 ways.
- 7 knights - in 12 ways.
- 6 knights - in 30 ways.

Case 4: 1 knight in the middle squares, such as

|  | A | $\mathrm{B}^{\prime}$ | X |
| :---: | :---: | :---: | :---: |
| B | Kn | $*$ | $\mathrm{~A}^{\prime}$ |
| $\mathrm{A}^{\prime}$ | $*$ | $*$ | X |
| X | $\mathrm{B}^{\prime}$ | X |  |

The two open squares are independent. Similar to Case 3.1, a knight on square A will attack a knight on square $\mathrm{A}^{\prime}$ and vice versa. However, both $\mathrm{A}^{\prime}$ squares are independent of each other, so they can both have a knight without attacking each other. The same is true for the B and $\mathrm{B}^{\prime}$ squares, respectively. We can place a maximum of 7 independent knights by placing knights on both $\mathrm{A}^{\prime}$ squares and both $B^{\prime}$ squares. Since the center knight can be on any of the center squares, there are 4 ways to get 7 independent knights. To get 6 knights we can either remove any one of the six outside knights, or remove both $\mathrm{A}^{\prime}$ (or both $\mathrm{B}^{\prime}$ ) knights and place a knight on A (or B). Thus there are $(6+2) \times 4=32$ ways to place 6 independent knights.

- 7 knights - in 4 ways.
- 6 knights - in 32 ways.

Case 5: 0 knights in the middle squares.

|  | A | $\mathrm{B}^{\prime}$ |  |
| :---: | :---: | :---: | :---: |
| B | $*$ | $*$ | $\mathrm{~A}^{\prime}$ |
| $\mathrm{A}^{\prime}$ | $*$ | $*$ | B |
|  | $\mathrm{~B}^{\prime}$ | A |  |

The four outer corner squares are independent.
Each A square attacks the A' squares, but not the other A, and vice versa. Similarly with the $\mathrm{B}, \mathrm{B}^{\prime}$ squares. We can place a maximum of 8 independent knights using all the corner squares and both A (or both $\mathrm{A}^{\prime}$ ) squares and both B (or $\mathrm{B}^{\prime}$ ) squares.

Thus there are 4 ways to place 8 independent knights. We can remove any one of the eight outer knights to get a configuration with 7 knights, so there are $8 \times 4=32$ configurations with 7 independent knights.

Because removing two knights from different configurations, for example either two A or two $\mathrm{A}^{\prime}$, may leave the same configuration of 6 knights, we consider different cases.

- Place all four corner knights and either both A (or $\mathrm{A}^{\prime}$ or B or $\mathrm{B}^{\prime}$ ) knights, or one of A or $\mathrm{A}^{\prime}$ and one of B or $\mathrm{B}^{\prime}$. This gives $4+16=20$ configurations.
- Place any three of the corner knights (four choices) and place both of A (or $\mathrm{A}^{\prime}$ or B or $\mathrm{B}^{\prime}$ ) and any one of the four other letter (16 choices). This gives $4 \times 16=64$ configurations.
- Place any two of the corner knights (six ways) and place both A or $\mathrm{A}^{\prime}$ and both B or $\mathrm{B}^{\prime}$ squares (four choices). This gives $6 \times 4=24$ configurations.
Hence, there are $20+64+24=108$ configurations with 6 independent knights.
- 8 knights - in 4 ways.
- 7 knights - in 32 ways.
- 6 knights - in 108 ways.

Final answers:
(1) There are $30+32+108=170$ ways to place 6 independent knights
(2) There are $2+4=6$ ways to place 8 independent knights
(3) There are $12+4+32=48$ ways to place 7 independent knights
(4) In none of the cases was it possible to place more than 8 independent knights
\#1377: Proposed by George Stoica, Saint John, New Brunswick, Canada. Consider a decreasing and positive sequence $\left\{a_{2 k-1}\right\}_{k \geq 1}$ whose sum converges. Prove there exists another decreasing and positive sequence $\left\{a_{2 k}\right\}_{k \geq 1}$ such that the combined sequence $\left\{a_{n}\right\}_{n \geq 1}$ is decreasing and the series $\sum_{n=1}^{\infty} a_{n}$ converges to an irrational number.
Solved by Levent Batakci (Case Western Reserve University). Denote the sum of the odd indices by $S$ :

$$
S:=\sum_{n=1}^{\infty} a_{2 n-1} .
$$

Choose $x \in \mathbb{Q}$ such that $2 S-\frac{a_{1}}{2}<x<2 S-\frac{a_{3}}{2}$; we can do this since the rationals are dense.
Define the sequence

$$
\left\{a_{2 n}\right\}:=\left\{x-\left(2 S-\frac{a_{1}}{2}\right)+\frac{a_{1}+a_{3}}{2}, \frac{a_{3}+a_{5}}{2}, \ldots, \frac{a_{2 n-1}+a_{2 n+1}}{2}, \ldots\right\} .
$$

It can easily be verified that the sum of these terms is

$$
\sum_{n=1}^{\infty} a_{2 n}=x-S
$$

We have that $\left\{a_{n}\right\}$ is decreasing and $\sum_{n=1}^{\infty} a_{n}=\sum_{n=1}^{\infty} a_{2 n-1}+\sum_{n=1}^{\infty} a_{2 n}=S+(x-S)=x$, which is rational.
\#1380: Proposed by George Stoica, Saint John, New Brunswick, Canada. It is very well
known that, if $\lim _{n \rightarrow \infty} \sum_{i=1}^{n} a_{i}$ exists, then $\lim _{n \rightarrow \infty} a_{n}=0$. Prove that the conclusion may be false under the (weaker) hypothesis that $\lim _{n \rightarrow \infty} \sum_{i=[n / 2]+1}^{n} a_{i}$ exists. (Square brackets denote the integer part).
Solved by Sarah Westmoreland and Frank Patane (Stamford University). We prove the conclusion is false by exhibiting a sequence $\left\{a_{n}\right\}$ that has both $a_{n} \nrightarrow 0$ and $\lim _{n \rightarrow \infty} \sum_{i=[n / 2]+1}^{n} a_{i}$ exists.

Let $a_{n}=0$ for all integers $n \geq 1$ except in the case when $n$ is a power of 2 , and in this case we define $a_{2^{k}}=1$. The sequence $\left\{a_{n}\right\}$ satisfies $\lim _{n \rightarrow \infty} a_{n} \neq 0$ and we claim $\sum_{i=[n / 2]+1}^{n} a_{i}=1$ for all $n$. This claim follows from the fact that there is a unique integer of the form $2^{k}$ in the set $I_{n}=\left\{\left[\frac{n}{2}\right]+1, \ldots, n\right\}$. If $n$ is of the form $n=2^{k}$ it is not hard to see the claim holds. If $n$ is not of the form $2^{k}$ then there exists a unique $k$ with

$$
2^{k} \leq n<2^{k+1}
$$

which implies

$$
2^{k-1} \leq[n / 2]+1<2^{k} \leq n
$$

and thus we again see there is a unique integer of the form $2^{k}$ in the required interval. Hence the sum $\sum_{i=[n / 2]+1}^{n} a_{i}$ equals 1 for all $n$, and thus the conclusion of the stated problem is shown to be false under the weaker hypothesis.
\#1381: Proposed by George Stoica, Saint John, New Brunswick, Canada. Is it true that $x_{n} \rightarrow 0$ if $x_{n+1}=\left|x_{n}-a_{n}\right|$ for all $n \geq 1$, with $a_{n} \searrow 0$ and $\sum_{n=1}^{\infty} a_{n}=\infty$ ?
Solved by Levent Batakci (Case Western Reserve University). Yes, it is true that $x_{n} \rightarrow 0$ if $x_{n+1}=\left|x_{n}-a_{n}\right|$ for all $n \geq 1$, with $a_{n} \searrow 0$ and $\sum_{n=1}^{\infty} a_{n}=\infty$.

By definition, $x_{n}$ is non-negative for all $n \geq 2$. Thus, it suffices to consider the sequence from index 2 onward. We claim that there must be infinitely many $m \in \mathbb{N}$ such that $x_{m} \leq a_{m}$. For the sake of contradiction, assume that this is not the case.

Case 1: There is no $m$ for which $x_{m} \leq a_{m}$. Then, we get that

$$
x_{j+1}=x_{j}-a_{j}=\left(x_{j-1}-a_{j-1}\right)-a_{j}=\cdots=x_{1}-\sum_{n=1}^{j} a_{n} \geq 0
$$

for all $j$. This implies that $\sum_{n=1}^{\infty} a_{n} \leq x_{1}$, which contradicts the hypotheses.

Case 2: There are only finitely many $m$ for which $x_{m} \leq a_{m}$. Let $M=1+$ $\max \left\{m \mid x_{m} \leq a_{m}\right\}$. Then, for all $j \geq M$, we have

$$
x_{j+1}=x_{M}-\sum_{n=M}^{j} a_{n} \geq 0 .
$$

This implies $\sum_{n=M}^{\infty} a_{n} \leq x_{M}$, which contradicts the hypotheses since the tail of a series diverging to infinity must also diverge to infinity.

Thus, there must be infinitely many $m \in \mathbb{N}$ for which $x_{m} \leq a_{m}$. Inductively, we will show that if $x_{m} \leq a_{m}$, then $x_{n} \leq a_{m}$ for all $n \geq m$.

Note that since $a_{n} \searrow 0$ (converges to 0 from above), we know $a_{n}$ must be non-negative for all $n$. Furthermore, $x_{n} \geq 0$ for all $n \geq 2$. We use of the fact that

$$
\begin{equation*}
|a-b| \leq \max \{a, b\} \text { for any } a, b \geq 0 \tag{2.1}
\end{equation*}
$$

Let $m$ be an index such that $x_{m} \leq a_{m}$. By (2.1), $x_{m+1}=\left|x_{m}-a_{m}\right| \leq \max \left\{x_{m}, a_{m}\right\}=a_{m}$. This establishes the base case.

Assume $x_{m+k} \leq a_{m}$ for some $k \in \mathbb{N}$. Since $\left\{a_{n}\right\}$ is decreasing,

$$
x_{m+k+1}=\left|x_{m+k}-a_{m+k}\right| \leq \max \left\{x_{m+k}, a_{m+k}\right\} \leq \max \left\{x_{m+k}, a_{m}\right\} \leq a_{m}
$$

proving the inductive step and completing the proof.
Thus, we've shown that

$$
\begin{equation*}
x_{m} \leq a_{m} \Rightarrow x_{j} \leq a_{m} \text { for all } j \geq m \tag{2.2}
\end{equation*}
$$

For the sake of contradiction, assume that $x_{n} \nrightarrow 0$. Then, there is an $\epsilon>0$ such that $x_{n}>\epsilon$ infinitely many times. Since $a_{n} \searrow 0$, there is some $M$ such that $a_{m} \leq \epsilon$ for all $m \geq M$. However, by (2.2), this would imply we have no $m \geq M$ such that $x_{m} \leq a_{m}$, contradicting that there must be infinitely many such $m$. Thus, we conclude that $x_{n} \rightarrow 0$.

GRE Practice \#9: The following is Problem \#14 from https://www.ets.org/s/gre/ pdf/practice_book_math.pdf: Suppose $g$ is a continuous real-valued function such that

$$
3 x^{5}+96=\int_{c}^{x} g(t) d t
$$

for each $x \in \mathbb{R}$, where $c$ is a constant. What is the value of $c$ ?
(a) -96
(b) -2
(c) 4
(d) 15
(e) 32 .

A painful way to solve this, or at least one that would take a bit of time, is to find $g(t)$. If we differentiate both sides, we find $15 x^{4}=g(x)$. We then integrate $g$ and find $3 x^{5}+96=3 x^{5}-3 c^{5}$, so $96=-3 c^{5}$ or $c=-2$. A faster way is to notice that this relation must hold for all $x$, and thus perhaps there is a good choice of $x$. If we try $x=c$ then the integral is just zero, and we find $3 c^{5}+96=0$, and again obtain $c=-2$. This problem illustrates an important item to remember for GRE problems - you only need to discover enough to answer the question asked. There is no need to find $g$; if finding $g$ helps you solve the problem that is great, but the goal is to find $c$.

Email address: sjm1@williams.edu
Professor of Mathematics, Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

