# PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2022 

STEVEN J. MILLER (EDITOR)

## 1. Problems: Fall 2022

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk $\left({ }^{*}\right)$ preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $\$ 100$ ). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). The winning school from the Fall problem set is Christopher Newport University.
\#1388: Proposed by Kenneth Davenport. We place the numbers that are 1 modulo 8 in a matrix as follows: we start with 1 in the upper left corner, then in the next diagonal put 9

Date: November 17, 2022.


Figure 1. Pizza motivation; can you name the theorem that's represented here?
then 17 , in the next diagonal 25,33 , and 41 , and so on. We show its start:

$$
\left(\begin{array}{cccccc}
1 & 9 & 25 & 49 & 81 & \cdots \\
17 & 33 & 57 & 89 & 129 & \cdots \\
41 & 65 & 97 & 137 & 185 & \cdots \\
73 & 105 & 145 & 193 & 249 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
$$

Prove or disprove that the sum of any two consecutive diagonals, running upper right to bottom left, is a perfect cube. For example,

$$
(1)+(9+17)=27=3^{3}, \quad(9+17)+(25+33+41)=125=5^{3} .
$$

If you cannot prove or disprove the conjecture, for how many diagonals can you confirm it?
\#1389: Proposed by Ron Evans, UCSD. This is a recasting of a conundrum presented in a book by J. Stein's book. There are fixed points $a$ and $b$ on the real line with $a<b$. You are standing in the dead of night at either point $a$ or point $b$, but you don't know which; all you know is that either choice is equally likely. There is a water source at a point $c$ halfway between $a$ and $b$. If you head toward $c$ immediately, you'll make it just in time to avoid dying of thirst. Unfortunately, you are not sure whether to head east or west. One survival strategy is to head east. This succeeds if you happened to be starting at $a$, but if you were starting at $b$, you die. With this strategy, you have only a $50 \%$ chance of reaching $c$. All you have with you is a phone app which picks a random real number x (say normally distributed) and tells you which direction to go to reach $x$. This app is seemingly useless, because your goal is to head toward $c$ rather than toward some random $x$. Is there a strategy that gives you a greater than $50 \%$ chance of reaching $c$ ?
\#1390: Proposed by Ron Evans, UCSD. his is based on a problem posed in a book of J. Stein. For a fixed odd positive integer $n$, and $p \in[0,1]$, define

$$
A(p):=\sum_{k=(n+1) / 2}^{n}\binom{n}{k} p^{k}(1-p)^{n-k} .
$$

(If say we interpret $p$ say as the probability that a certain product passes the manufacturer's reliability test, then $A(p)$ equals the probability that after $n$ tests, there will be at least $(n+1) / 2$ successes, that is, more successes than failures.) PROBLEM: Let $x, y \in[0,1]$ with $x+y \geq 1$. Show that $A(x)+A(y)$ attains its minimum value if and only if $x+y=1$.
\#1391: Proposed by Seán M. Stewart, King Abdullah University of Science and Technology.

Evaluate

$$
\int_{0}^{\frac{\pi}{6}} \frac{x}{\sin ^{2} x \sqrt{\cot ^{2} x-3}} d x
$$

The integral arises as a particular case in the calculation of the differential cross section of an electron scattering from a heavy nucleus that is at rest. The interaction between the projectile and the target is considered to arise from a Coulomb potential. The general result is referred to as "the marvelous identity," is found using an indirect proof that relies on a physical argument, and whose validity is confirmed using numerical integration.

Cross section is a measure of the probability an incoming projectile will be scattered or deflected through a given angle during a collision with the target. It is a stochastic process. When expressed as the differential limit of a function of the projectile's angle (or some other final-state variable such as energy), the cross section is referred to as a differential cross section.
\#1392: Proposed by George Jennings, David Ni, Wai Yan Pong, and Serban Raianu, California State University, Dominguez Hills. Show that the distance from a point on the hyperbola $x^{2}-y^{2}=1$ to the $x$-axis is equal to the length of the tangent from the projection of the point to the $x$-axis to the circle $x^{2}+y^{2}=1$; see Figure 2 .


Figure 2. Circle-hyperbola from Problem \#1392.
\#1393: Proposed by Toth Attila. Let $p>2$ be a prime. Prove that if there are positive integers $x, y$ and $z$ with $x^{p}+y^{p}=z^{p}$ then $p$ must divide $x+y-z$. Note: observations such as this can help winnow the list of candidate solutions which need to be investigated.

GRE Practice \#10: What is the value of $\int_{-\pi / 4}^{\pi / 4}\left(\cos t+\sqrt{1+t^{2}} \sin ^{3} t \cos ^{3} t\right) d t$ ? Question from https://tinyurl.com/d3ynvtbr.
(a) 0
(b) $\sqrt{2}$
(c) $\sqrt{2}-1$
(d) $\sqrt{2} / 2$
(e) $(\sqrt{2}-1) / 2$.

## 2. Solutions

\#1382: Proposed by George Stoica, Saint John, New Brunswick, Canada. Find all integers $a, b, c, d$ such that $a^{3}+b^{3}+c^{3}+d^{3}=2021$; note the integers may be zero or negative. Due to a backlog this problem is appearing in 2022 instead of 2021; if you wish, consider instead $1^{3}+a^{3}+b^{3}+c^{3}+d^{3}=2022$.

## Solution below by Ashland University Problem Solving Group.

Below is a partial solution. We looked for any pattern suggesting that there is a limit to the number of solutions to this problem but have not yet found one. However, we have found that there are six unique solutions to this problem where and and. The solutions are (83, -19, -64, -67), (68, -10, -22, -67), (77, 14, -58, -64), (47, 41, 29, -58), (47, -7, -28, -43), and (38, -1, -13, -37). We know that at least one value must be positive and at least one value must be negative since there is no sum of four non-negative cubes that equal 2021.

Note from editor: As we can have positive and negative values cancel, it is not necessarily a bounded search. For more on challenges of such problems, see for example https: // tinyurl. com/2fkwzzvv.
\#1383: Proposed by Kenneth Davenport. Consider the following collection of numbers:

$$
(1), \quad\left(\begin{array}{ll}
1 & 3 \\
5 & 9
\end{array}\right), \quad\left(\begin{array}{ccc}
1 & 3 & 7 \\
5 & 9 & 15 \\
11 & 17 & 25
\end{array}\right), \ldots
$$

that is to say in the $n \times n$ matrix we have $a_{i j}$, the entry in the $i^{\text {th }}$ row and $j^{\text {th }}$ column, equals $j^{2}+(2 i-3) j+\left(i^{2}-i+1\right)$.
(a) For what values of $n$ is the sum of the elements of the $n \times n$ matrix a square? For example, if $n=25$ the sum is $18255=135^{2}$.
(b) Consider the sum of the elements on the non-main diagonal:

$$
\begin{aligned}
1 & =1^{3} \\
3+5 & =2^{3} \\
7+9+11 & =3^{3} \\
13+15+17+19 & =4^{3} .
\end{aligned}
$$

Does this pattern continue? If yes prove it; if not, why not (finding a counter-example suffices).
Solution below by Brian Bradie, Christopher Newport University. Also solved by Soham Dutta, DPS Ruby Park High School.

Consider the following collection of numbers:

$$
(1), \quad\left(\begin{array}{cc}
1 & 3 \\
5 & 9
\end{array}\right), \quad\left(\begin{array}{ccc}
1 & 3 & 7 \\
5 & 9 & 15 \\
11 & 17 & 25
\end{array}\right), \quad \ldots
$$

that is to say in the $n \times n$ matrix we have $a_{i j}$, the entry in the $i$ th row and $j$ th column, equals $j^{2}+(2 i-3) j+\left(i^{2}-i+1\right)$.
(a) For what values of $n$ is the sum of the elements of the $n \times n$ matrix a square? For example, if $n=25$ the sum is $18255=135^{2}$.
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\begin{aligned}
& 1=1^{3} \\
& 3+5=2^{3} \\
& 7+9+11=3^{3} \\
& 13+15+17+19=4^{3} .
\end{aligned}
$$

Does this pattern continue? If yes, prove it; if not, why not (finding a counterexample suffices).

## Solution:

(a) The sum of the elements in the $n \times n$ matrix is

$$
\begin{aligned}
\sum_{i=1}^{n} & \sum_{j=1}^{n}\left(j^{2}+(2 i-3) j+\left(i^{2}-i+1\right)\right) \\
& =\sum_{i=1}^{n}\left(\frac{n(n+1)(2 n+1)}{6}+(2 i-3) \frac{n(n+1)}{2}+n\left(i^{2}-i+1\right)\right) \\
& =\frac{n^{2}(n+1)(2 n+1)}{6}+(n(n+1)-3 n) \frac{n(n+1)}{2}+n\left(\frac{n(n+1)(2 n+1)}{6}-\frac{n(n+1)}{2}+n\right) \\
& =n^{2}\left(\frac{7 n^{2}-1}{6}\right)
\end{aligned}
$$

Therefore, the sum of the elements in the $n \times n$ matrix is square when $\frac{7 n^{2}-1}{6}$ is square; that is, when

$$
6 m^{2}-7 n^{2}=-1
$$

for some integer $m$. Multiplying this last equation by 6 yields

$$
x^{2}-42 n^{2}=-6
$$

where $x=6 \mathrm{~m}$. The first few convergents for $\sqrt{42}$ are

$$
6, \quad 6+\frac{1}{2}=\frac{13}{2}, \quad 6+\frac{1}{2+\frac{1}{12}}=\frac{162}{25}, \quad 6+\frac{1}{2+\frac{1}{12+\frac{1}{2}}}=\frac{337}{52}
$$

Because
$6^{2}-42(1)^{2}=162^{2}-42(25)^{2}=-6$ but $13^{2}-42(2)^{2}=337^{2}-42(52)^{2}=1$,
it follows that the integer solutions, $\left(x_{k}, n_{k}\right)$, of $x^{2}-42 n^{2}=-6$ are given by

$$
x_{k}+n_{k} \sqrt{42}=(6+\sqrt{42})^{2 k+1}=(6+\sqrt{42})(13+2 \sqrt{42})^{k}
$$

for each non-negative integer $k$. Values for $x_{k}$ and $n_{k}$ can be obtained via the coupled recurrences

$$
x_{k+1}=13 x_{k}+84 n_{k}, \quad n_{k+1}=2 x_{k}+13 n_{k}, \quad x_{0}=6, n_{0}=1
$$

or the uncoupled recurrences

$$
\begin{aligned}
& x_{k+1}=26 x_{k}-x_{k-1}, \quad x_{0}=6, x_{1}=162 \\
& n_{k+1}=26 n_{k}-n_{k-1}, \quad n_{0}=1, n_{1}=25
\end{aligned}
$$

Note 6 divides both $x_{0}$ and $x_{1}$, so applying induction through the uncoupled recurrence relation for $x_{k}$ established that 6 divides $x_{k}$ for each non-negative integer $k$. Thus, every integer solution to $x^{2}-42 n^{2}=-6$ corresponds to an integer solution to $6 m^{2}-7 n^{2}=-1$. An explicit solution to the uncoupled $n_{k}$ recurrence is

$$
n_{k}=\frac{(6+\sqrt{42})(13+2 \sqrt{42})^{k}-(6-\sqrt{42})(13-2 \sqrt{42})^{k}}{2 \sqrt{42}}
$$

Finally, the sum of the elements in the $n \times n$ matrix is square when

$$
n=\frac{(6+\sqrt{42})(13+2 \sqrt{42})^{k}-(6-\sqrt{42})(13-2 \sqrt{42})^{k}}{2 \sqrt{42}}
$$

for some non-negative integer $k$.
(b) For the elements along the non-main diagonal, $i+j=n+1$. Thus, the sum of the elements along the non-main diagonal is

$$
\begin{aligned}
\sum_{i=1}^{n}\left((n+1-i)^{2}+(2 i-3)(n+1-i)+\left(i^{2}-i+1\right)\right) & =\sum_{i=1}^{n}\left(n^{2}-n-1+2 i\right) \\
& =n\left(n^{2}-n-1\right)+n(n+1) \\
& =n^{3} .
\end{aligned}
$$

The pattern continues.
\#1385: Proposed by Hongwei Chen, Christopher Newport University, Newport News, Virginia. A continued fraction is an expression of the form

$$
a_{0}+\frac{1}{a_{1}+\frac{1}{a_{2}+\ldots}}
$$

where the $a_{i}$ are positive integers for all $i \in \mathbb{N}$. It is often denoted by $\left[a_{0} ; a_{1}, a_{2}, \ldots\right]$.
Continued fractions arise naturally in long division and in the theory of approximation to real numbers by rationals. It is known that every finite continued fraction represents a rational number and every periodic continued fraction represents an irrational root of a quadratic equation with integral coefficients. A famous example is $[1,1,1, \ldots]=[\overline{1}]=\phi$, the golden ratio.

This problem is inspired by the recent CMJ Problem 1186 (51:5, 386 (2020)):
Find a closed-form expression for the continued fraction $[1,1, \ldots, 1,3,1,1, \ldots, 1]$, which has $n$ ones before and after, the middle three.

We found the required closed-form expression is

$$
x_{n}:=\frac{F_{n+4} F_{n+1}}{F_{n+2}^{2}}
$$

where $F_{n}$ is the $n$th Fibonacci number.
Solution below by Emmanuel Adutwum, Aakash Gurung, Saea Eun Lee, and Johnathan Park, Juniata College.

Find a closed-form expression for the continued fraction $[1,1, \ldots, 1,3,1,1, \ldots, 1]$, which has $n$ ones before and after, the middle three. We found the required closed-form expression
is

$$
x_{n}=\frac{F_{n+4} F_{n+1}}{F_{n+2}^{2}}
$$

where $F_{n}$ is the $n$th Fibonacci number.
Proof. Consider the Fibonacci numbers and the flip-and-plus-one function $f(x):-1+\frac{1}{x}=$ $\frac{x+1}{x}$

| $F_{1}$ | $F_{2}$ | $F_{3}$ | $F_{4}$ | $F_{5}$ | $F_{6}$ | $F_{7}$ | $F_{8}$ | $F_{9}$ | $\cdots \cdots$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | $\cdots \cdots$ |

We define its $n$th iteration as $f^{(n)}(x)=f(f(f(\cdots f(x))))$ for every $n \in \mathbb{N}$. We calculated a few iterations and conjectured a formula as below.

Claim: For every $n \geq 2, f^{(n)}(x)=\frac{F_{n+1} x+F_{n}}{F_{n} x+F_{n-1}}$. Indeed, when $n=2$,

$$
f(f(x))=\frac{f(x)+1}{f(x)}=\frac{\frac{x+1}{x}+1}{\frac{x+1}{x}}=\frac{2 x+1}{x+1}=\frac{F_{3} x+F_{2}}{F_{2} x+F_{1}}
$$

Suppose $f^{(k)}(x)=\frac{F_{k+1} x+F_{k}}{F_{k} x+F_{k-1}}$ for every $k \geq 2$. We want to show that $f^{(k+1)}(x)=\frac{F_{k+2} x+F_{k+1}}{F_{k+1} x+F_{k}}$. Indeed,

$$
f^{(k+1)}(x)=f\left(f^{k}(x)\right)=\frac{\frac{F_{k+1} x+F_{k}}{F_{k} x+F_{k-1}}+1}{\frac{F_{k+1} x+F_{k}}{F_{k} x+F_{k-1}}}=\frac{F_{k+1} x+F_{k}+F_{k} x+F_{k-1}}{F_{k+1} x+F_{k}}=\frac{F_{k+2} x+F_{k+1}}{F_{k+1} x+F_{k}} .
$$

According to the definition of $x_{n}$, notice that $x_{n}$ can be formulated as follows:

$$
x_{n}=f^{(n)}\left(2+f^{(n)}(1)\right)
$$

It is quite a repeated work of distribution and definition of Fibonacci numbers to show that $f^{(n)}\left(2+f^{(n)}(1)\right)=\frac{F_{n+4}+F_{n+1}}{F_{n+2}^{2}}$. We will break it down it several steps.

$$
f^{(n)}\left(2+f^{(n)}(1)\right)=\frac{F_{n+1}\left(2+f^{(n)}(1)\right)+F_{n}}{F_{n}\left(2+f^{(n)}(1)\right)+F_{n-1}}=\frac{F_{n+1}\left(2+\frac{F_{n+1}+F_{n}}{F_{n}+F_{n-1}}\right)+F_{n}}{F_{n}\left(2+\frac{F_{n+1}+F_{n}}{F_{n}+F_{n-1}}\right)+F_{n-1}}=\frac{F_{n+1}\left(2+\frac{F_{n+2}}{F_{n+1}}\right)+F_{n}}{F_{n}\left(2+\frac{F_{n+2}}{F_{n+1}}\right)+F_{n-1}}
$$

Continuing with the numerator, we have

$$
F_{n+1}\left(2+\frac{F_{n+2}}{F_{n+1}}\right)+F_{n}=2 F_{n+1}+F_{n+2}+F_{n}=F_{n+1}+2 F_{n+2}=F_{n+2}+F_{n+3}=F_{n+4} .
$$

Continuing with the denominator, we have

$$
\begin{aligned}
F_{n}\left(2+\frac{F_{n+2}}{F_{n+1}}\right)+F_{n-1}=\frac{2 F_{n} F_{n+1}+F_{n} F_{n+2}+F_{n-1} F_{n+1}}{F_{n+1}} & =\frac{F_{n+1}\left(2 F_{n}+F_{n-1}\right)+F_{n} F_{n+2}}{F_{n+1}} \\
& =\frac{F_{n+1}\left(F_{n+2}\right)+F_{n} F_{n+2}}{F_{n+1}} \\
& =\frac{F_{n+2}\left(F_{n+1}+F_{n}\right)}{F_{n+1}}=\frac{F_{n+2}^{2}}{F_{n+1}}
\end{aligned}
$$

Therefore our result does match the desired closed-form expression.

$$
x_{n}=f^{(n)}\left(2+f^{(n)}(1)\right)=\frac{F_{n+4}}{\frac{F_{n+2}^{2}}{F_{n+1}}}=\frac{F_{n+4} F_{n+1}}{F_{n+2}^{2}} .
$$

\#1387: Proposed by Hongbiao Zeng, Fort Hays State University, Hays, KS. Show that for any positive integer $n$, if $p$ is a prime number such that $n<p<2 n$, then $\binom{2 n}{p} \equiv 1(\bmod p)$. Solution below by Brian Bradie, Christopher Newport University. Also solved by Soham Dutta, DPS Ruby Park High School.

Write

$$
\binom{2 n}{p}=\frac{(2 n)(2 n-1)(2 n-2) \cdots(p+1)}{(2 n-p)!}
$$

Because $n<p<2 n$, it follows that $2 n<2 p$, so none of the factors

$$
2 n, \quad 2 n-1, \quad 2 n-2, \quad \cdots \quad p+1
$$

is divisible by $p$. Moreover, $0<2 n-p<n<p$ and

$$
(p+j) \equiv j \quad(\bmod p)
$$

So

$$
\begin{aligned}
(2 n)(2 n-1)(2 n & -2) \\
& \cdots(p+1) \\
& =(p+2 n-p)(p+2 n-1-p)(p+2 n-2-p) \cdots(p+1) \\
& \equiv(2 n-p)!(\bmod p)
\end{aligned}
$$

which is not congruent to zero modulo $p$. Finally,

$$
\binom{2 n}{p}=\frac{(2 n)(2 n-1)(2 n-2) \cdots(p+1)}{(2 n-p)!} \equiv 1 \quad(\bmod p)
$$

GRE Practice \#10: What is the value of $\int_{-\pi / 4}^{\pi / 4}\left(\cos t+\sqrt{1+t^{2}} \sin ^{3} t \cos ^{3} t\right) d t$ ? Question from https://tinyurl.com/d3ynvtbr.
(a) 0
(b) $\sqrt{2}$
(c) $\sqrt{2}-1$
(d) $\sqrt{2} / 2$
(e) $(\sqrt{2}-1) / 2$.

Solution: While it is natural to try to find an anti-derivative, we quickly see that such an approach will not work here, as the second summand is a nightmare, involving a square-root and products of cubes of trig functions. Fortunately we can ignore this, as that factor is an odd function $(f(-x)=-f(x))$ and we are integrating over a region symmetric about zero, so its integral is zero. Thus we are left with $\int_{-\pi / 4}^{\pi / 4} \cos t d t$. We could integrate it directly, noting the anti-derivative of cosine is sine; doing so gives $\sin (\pi / 4)-\sin (-\pi / 4)=\sqrt{2}$, so the answer is (b).

While not needed for this problem, we comment on some other arguments one could use to eliminate some answers. Since cosine is an even function $(f(-x)=f(x))$, the integral is just $2 \int_{0}^{\pi / 4} \cos t d t$. As $0 \leq \cos t \leq 1$ in this interval, the answer has to be between 0 and 2 .

Further, $\cos (0)=1$ and the function $\cos t$ is decreasing in the interval, with smallest value at $\pi / 4$ where it is $\sqrt{2} / 2$. Thus

$$
\frac{\pi \sqrt{2}}{4}=2 \cdot \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} \leq 2 \int_{0}^{\pi / 4} \cos t d t \leq 2 \cdot \frac{\pi}{4} \cdot 1=\frac{\pi}{2}
$$

Unfortunately the upper bound exceeds all the answers for this problem, and is of no help. The lower bound however helps; it is larger than $\sqrt{2} / 2$ so we can eliminate (a), (d) and (e), and a little inspection shows the lower bound is also larger than $\sqrt{2}-1$, eliminating (c). Thus the answer has to be (b), and we are able to see this without actually doing the integration!

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