PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2023

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2023

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). The winning school from the Fall problem set is **Christopher Newport University**.

#1394: *Proposed by Steven Miller, Williams College.* When we dropped my son off at Camp Winadu we received a blue and white half moon (those are his camp's colors). A half moon is a delicious frosted cookie, with frosting of one color on half of the circle and another color on the other. Splitting it in two is easy to do so that everyone gets an equal amount of the two colors. What is a good way to split it into three equal parts where each has the same amount of each color? Describe exactly where you make the cuts, assume all you can do is

Date: July 21, 2023.

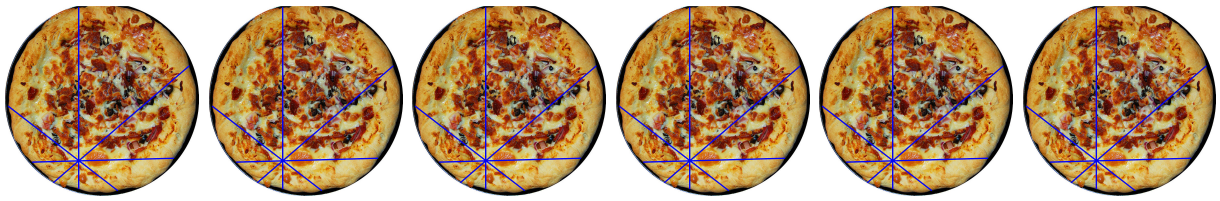

FIGURE 1. Pizza motivation; can you name the theorem that's represented here?

cut in any straight line. (Note of course it is trivial if you do not care about the amount of each color; choose 3 equi-spaced points on the perimeter and cut from the center to these.)

#1395: *Proposed by Hongwei Chen, Christopher Newport University.* Show that the Fourier sine series of $\ln(\tan x)$ on $(0, \pi/2)$ is given by

$$\ln(\tan x) \;=\; -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{h_n}{n}\sin(4nx), \tag{1}$$

where

$$h_n \;=\; 1+\frac{1}{3}+\cdots+\frac{1}{2n-1}.$$

Motivation: It is well-known that $\{\cos(2nx)\}_{n=0}^{\infty}$ forms an orthogonal basis of $L^2(0,\pi/2)$. In particular, we have

$$\begin{aligned} -\ln(\sin x) &= \ln 2+\sum_{k=1}^{\infty}\frac{1}{k}\cos(2kx); \\ -\ln(\cos x) &= \ln 2+\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\cos(2kx). \end{aligned}$$

This yields the Fourier cosine series of $\ln(\tan x)$:

$$-\ln(\tan x) \;=\; 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\cos(2(2k+1)x), \qquad x\in(0,\pi/2). \tag{2}$$

It is natural to ask for its corresponding Fourier sine series. Since

$$h_n \;=\; H_{2n}-\frac{1}{2}H_n,$$

where H_n is the nth harmonic number, our formula enables us to find some exact values of Euler sums. For example, applying Parseval's identity, together with

$$\int_0^{\pi/2}\ln^2(\tan x)\,dx \;=\; \frac{1}{8}\pi^3,$$

we recover the identity

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\,h_n^2 \;=\; \frac{1}{32}\pi^4.$$

#1396: *Proposed by Suaib Lateef, Oke oore, Iwo, Osun State, Nigeria.*

Let

$$f_{(n,p)} \;=\; \sum_{k=0}^{n} \binom{n}{k}^{p},$$

where $\binom{n}{k}$ is the standard binomial coefficient $\frac{n!}{k!(n-k)!}$. In 1894, Franel[1] showed that

$$(n+1)^2 f_{(n+1,3)} \;=\; (7n^2+7n+2)f_{(n,3)} + 8n^2 f_{(n-1,3)},$$

which can be re-written as

$$f_{(n,3)} \;=\; \frac{(7n^2-7n+2)f_{(n-1,3)} + 8(n-1)^2 f_{(n-2,3)}}{n^2}, \tag{1}$$

for $n > 1$. Also in 1895, Franel[2] proved that

$$(n+1)^3 f_{(n+1,4)} \;=\; 2(2n+1)(3n^2+3n+1)f_{(n,4)} + 4(4n-1)(4n+1)f_{(n-1,4)},$$

which can also be re-written as

$$f_{(n,4)} \;=\; \frac{2(2n-1)(3n^2-3n+1)f_{(n-1,4)} + 4(4n-5)(4n-3)f_{(n-2,4)}}{n^3}, \tag{2}$$

for $n > 1$.

V. Strehl[3] in1994 showed that

$$f_{(n,3)} \;=\; \sum_{k=0}^{n} \binom{n}{k}^{2} \binom{2k}{n}. \tag{3}$$

We can see that the right-hand sides of (1) and (3) are two different expressions for $f_{(n,3)}$. One could be curious to know if there are many other expressions for $f_{(n,3)}$ and possibly $f_{(n,p)}$, for all real and complex p. This curiosity leads us to ask the following question:
If p is any real or complex number, n is any positive integer, $\binom{n}{k}$ is a Binomial coefficient, and $\pm a_2$, $\pm a_3$, $\pm a_4 ... \pm a_r$ are some integers, does

$$\sum_{k=0}^{n} \binom{n}{k}^{p} \left(1 + \sum_{j=2}^{r} \pm a_j \left(\frac{k}{n}\right)^{j}\right) = 0$$

exist for all $r \geq 2$?

As a start, prove

$$\sum_{k=0}^{n} \binom{n}{k}^{3} \;=\; 6\sum_{k=0}^{n-1} \binom{n}{k+1}\binom{n-1}{k}^{2} - 4\sum_{k=0}^{n-1} \binom{n-1}{k}^{3}.$$

[1]J. Franel, *On a question of Laisant*, L'intermiaire des Mathaticiens, **1** (1894), 45–47.

[2]J. Franel, *On a question of Laisant*, L'intermiaire des Mathaticiens, **2** (1895), 33–35.

[3]V. Strehl, *Binomial Identities-combinatorial and algorithmic aspects*, Discrete Math., **136** (1994), 309–346.

#1397: *Proposed by Suaib Lateef, Oke oore, Iwo, Osun State, Nigeria.* Prove

$$\sum_{k=0}^{n} \binom{n}{k}^p \left(1 - 6\left(\frac{k}{n}\right)^2 + 4\left(\frac{k}{n}\right)^3\right) \;=\; 0$$

and

$$\sum_{k=0}^{n} \binom{n}{k}^p \left(1 - 4\left(\frac{k}{n}\right)^2 - 4\left(\frac{k}{n}\right)^3 + 10\left(\frac{k}{n}\right)^4 - 4\left(\frac{k}{n}\right)^5\right) \;=\; 0.$$

#1398: *Proposed by Carsten Botts (Johns Hopkins University, Applied Physics Lab) and Steven J. Miller (Williams College).* Let $p(x)$ be a continuous probability distribution (so it is non-negative and integrates to 1) such that the logarithm of $p(x)$ is three times continuously differentiable. Construct such a function with infinitely many points of inflection. *Note: some people use inflection point to mean a point where the second derivative vanishes, while others use it to be a point where the function changes from concave to convex; if the third derivative is non-zero at the point then these two definitions are equivalent.*

#1399: *Proposed by Zhongxue Lü (Jiangsu Normal University) and Steven J. Miller (Williams College).* This problem is inspired from an observation in 2021 (due to the backlog of problems it is only being published now), where 2021 is formed by writing two consecutive integers one after the other; in other words it is of form $n \cdot 10^k + (n+1)$ where k is the number of digits of n and n has leading digit non-zero and is not all 9's. We call such integers 2-adjacent joined numbers. Note we do not consider 102 or 10000 such numbers (even though the first could be written as $01 * 10^2 + 02$ and the second as $99 * 10^2 + 100$). How many 2-adjacent joined numbers are there less than 10^{100}?

GRE Practice #11:
Let $f(x) = \sum_{n=1}^{\infty} x^n/n$ for $-1 < x < 1$. Then $f'(x)$ equals
(a) $\frac{1}{1-x}$ (b) $\frac{x}{1-x}$ (c) $\frac{1}{1+x}$ (d) $\frac{x}{1+x}$ (e) 0.

2. Solutions

Note: After the Fall 2022 issue went to press, we received some additional correct solutions and want to acknowledge the authors: Hyunbin Yoo from South Korea and Jeffrey Hemmelgarn at North Central College solved #1383 while his colleague Brennan Sweeney (also of North Central College) got Problem, #1387, while Sohom Dutta, DPS Ruby Park High School, Ian McDowell, University of South Carolina, Brian Bradie, Christopher Newport University, Hyunbin Yoo from South Korea and Rohan Dalal, The Episcopal Academy, the Eagle Problem Solvers of Georgia Southern University, the Cal Poly Pomona Problem Solving Group, and Thomas Reinke of Samford University all solved Problem #1388.

#1388: *Proposed by Kenneth Davenport.* We place the numbers that are 1 modulo 8 in a matrix as follows: we start with 1 in the upper left corner, then in the next diagonal put 9

then 17, in the next diagonal 25, 33, and 41, and so on. We show its start:

$$\begin{pmatrix} 1 & 9 & 25 & 49 & 81 & \cdots \\ 17 & 33 & 57 & 89 & 129 & \cdots \\ 41 & 65 & 97 & 137 & 185 & \cdots \\ 73 & 105 & 145 & 193 & 249 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Prove or disprove that the sum of any two consecutive diagonals, running upper right to bottom left, is a perfect cube. For example,

$$(1) + (9 + 17) \;=\; 27 \;=\; 3^3, \qquad (9 + 17) + (25 + 33 + 41) \;=\; 125 \;=\; 5^3.$$

If you cannot prove or disprove the conjecture, for how many diagonals can you confirm it?

Solution by Samuel Aguilar and the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA. The conjecture is true.

Let $a_n = 8n + 1$ for each nonnegative integer n, let d_n be the sum of the numbers in the nth diagonal, starting with $n = 0$, and let $T_n = n(n + 1)/2$ be the nth triangular number. Then $d_0 = a_0$, $d_1 = a_1 + a_2$, $d_2 = a_3 + a_4 + a_5$, and in general, for integers $n \geq 1$,

$$d_n = \sum_{k=T_n}^{T_{n+1}-1} a_k = \sum_{k=T_n}^{T_{n+1}-1} 8k + 1.$$

Then for any positive integer n, the sum of the numbers in the nth and $(n + 1)$st diagonals is given by

$$\begin{aligned}
d_n + d_{n+1} &= \sum_{k=T_n}^{T_{n+2}-1} 8k + 1 \\
&= 8 \sum_{k=T_n}^{T_{n+2}-1} k + \sum_{k=T_n}^{T_{n+2}-1} 1 \\
&= 8 \left(\frac{T_{n+2}(T_{n+2} - 1)}{2} - \frac{T_n(T_n - 1)}{2} \right) + T_{n+2} - 1 - (T_n - 1) \\
&= 4 \left(T_{n+2}^2 - T_{n+2} - T_n^2 + T_n \right) + T_{n+2} - T_n \\
&= 4 \left(T_{n+2} - T_n \right) \left(T_{n+2} + T_n - 1 \right) + T_{n+2} - T_n \\
&= \left(T_{n+2} - T_n \right) \left(4T_{n+2} + 4T_n - 3 \right) \\
&= \left(\frac{(n + 2)(n + 3)}{2} - \frac{n(n + 1)}{2} \right) \left(2(n + 2)(n + 3) + 2n(n + 1) - 3 \right) \\
&= \frac{1}{2} \left(n^2 + 5n + 6 - (n^2 + n) \right) \left(2n^2 + 10n + 12 + 2n^2 + 2n - 3 \right) \\
&= (2n + 3) \left(4n^2 + 12n + 9 \right) \\
&= (2n + 3)^3.
\end{aligned}$$

which is an odd perfect cube.

#1391: *Proposed by Seán M. Stewart, King Abdullah University of Science and Technology.* Evaluate

$$\int_0^{\frac{\pi}{6}} \frac{x}{\sin^2 x\sqrt{\cot^2 x - 3}}\, dx.$$

The integral arises as a particular case in the calculation of the *differential cross section* of an electron scattering from a heavy nucleus that is at rest. The interaction between the projectile and the target is considered to arise from a Coulomb potential. The general result is referred to as "the marvelous identity," is found using an indirect proof that relies on a physical argument, and whose validity is confirmed using numerical integration.

Cross section is a measure of the probability an incoming projectile will be scattered or deflected through a given angle during a collision with the target. It is a stochastic process. When expressed as the differential limit of a function of the projectile's angle (or some other final-state variable such as energy), the cross section is referred to as a differential cross section.

Solution below by ***Brian Bradie, Christopher Newport University****.* Consider the more general problem: evaluate

$$\int_0^{\cot^{-1}\sqrt{a}} \frac{x}{\sin^2 x\sqrt{\cot^2 x - a}}\, dx,$$

where $a > 0$. The original problem corresponds to the case $a = 3$.

Now, the substitution $u = \cot x$ followed by the substitution $u = \sqrt{a}\sec\theta$ and then using the identity

$$\tan^{-1}\frac{1}{x} \;=\; \cot^{-1} x$$

yields

$$\begin{aligned}
\int_0^{\cot^{-1}\sqrt{a}} \frac{x}{\sin^2 x\sqrt{\cot^2 x - a}}\, dx &= \int_{\sqrt{a}}^{\infty} \frac{\cot^{-1} u}{\sqrt{u^2 - a}}\, du \\
&= \int_0^{\pi/2} \sec\theta \cot^{-1}\left(\sqrt{a}\sec\theta\right)\, d\theta \\
&= \int_0^{\pi/2} \frac{\tan^{-1}\left(\frac{1}{\sqrt{a}}\cos\theta\right)}{\cos\theta}\, d\theta.
\end{aligned}$$

Let

$$I(\alpha) \;:=\; \int_0^{\pi/2} \frac{\tan^{-1}(\alpha\cos\theta)}{\cos\theta}\, d\theta.$$

Then,

$$\begin{aligned} I'(\alpha) &= \int_0^{\pi/2} \frac{1}{1+\alpha^2\cos^2\theta}\,d\theta = \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^2\theta+\alpha^2}\,d\theta \\ &= \int_0^{\pi/2} \frac{\sec^2\theta}{1+\alpha^2+\tan^2\theta}\,d\theta \\ &= \frac{1}{\sqrt{1+\alpha^2}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{1+\alpha^2}}\right)\Bigg|_0^{\pi/2} \\ &= \frac{\pi}{2\sqrt{1+\alpha^2}}. \end{aligned}$$

With $I(0)=0$, it then follows that

$$I(\alpha) = \frac{\pi}{2}\sinh^{-1}\alpha,$$

and

$$\int_0^{\cot^{-1}\sqrt{a}} \frac{x}{\sin^2 x\sqrt{\cot^2 x - a}}\,dx = I\left(\frac{1}{\sqrt{a}}\right) = \frac{\pi}{2}\sinh^{-1}\left(\frac{1}{\sqrt{a}}\right).$$

In particular,

$$\begin{aligned} \int_0^{\pi/6} \frac{x}{\sin^2 x\sqrt{\cot^2 x - 3}}\,dx &= \frac{\pi}{2}\sinh^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{\pi}{2}\ln\left(\frac{1}{\sqrt{3}}+\sqrt{1+\frac{1}{3}}\right) \\ &= \frac{\pi}{2}\ln\sqrt{3} = \frac{\pi}{4}\ln 3. \end{aligned}$$

#1392: *Proposed by George Jennings, David Ni, Wai Yan Pong, and Serban Raianu, California State University, Dominguez Hills.* Show that the distance from a point on the hyperbola $x^2-y^2=1$ to the x-axis is equal to the length of the tangent from the projection of the point to the x-axis to the circle $x^2+y^2=1$; see Figure 2.

Solution below by ***Soham Dutta, DPS Ruby Park High School****. Also solved by* ***Brian Bradie, Christopher Newport University****,* ***Nago Asahi, Aakash Gurung, and Xuan Pham, Juniata College****, the* ***Eagle Problem Solvers of Georgia Southern University*** *and the* ***Cal Poly Pomona Problem Solving Group****.*

Consider the hyperbola $x^2-y^2=1$. We can parametrize the points on it by $A(\theta)=(\sec\theta,\tan\theta)$, as $\sec^2\theta=\tan^2\theta+1$. Its projection point on the x-axis is $A'=(\sec\theta,0)$. Hence the distance of A from A' is $\tan\theta$ i.e $AA'=\tan\theta$.

Now consider the equation of a line ℓ passing through A' and tangent to the circle $x^2+y^2=1$. Let the point of contact of ℓ and the circle be C. Since $\angle OCA'=90^\circ$, by the Pythagorean

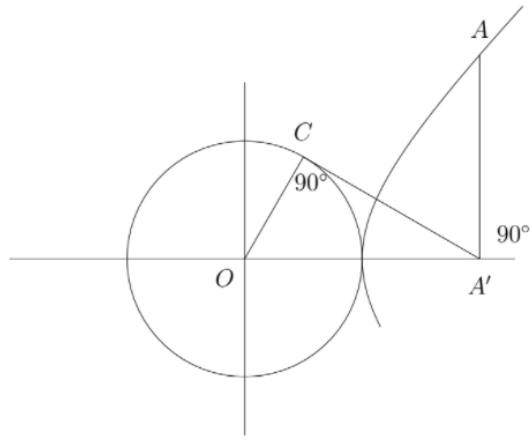

FIGURE 2. Circle-hyperbola from Problem #1392.

Theorem we have

$$\begin{aligned} OC^2 + CA'^2 &= OA'^2 \\ 1 + CA'^2 &= \sec^2\theta \\ CA'^2 &= \sec^2\theta - 1 = \tan^2\theta. \end{aligned}$$

Hence $CA' = \tan\theta = AA'$ which is exactly what we wanted to show!

#1393: *Proposed by Toth Attila.* Let $p > 2$ be a prime. Prove that if there are positive integers x, y and z with $x^p + y^p = z^p$ then p must divide $x + y - z$. *Note: observations such as this can help winnow the list of candidate solutions which need to be investigated.*
Solution below by the ***Eagle Problem Solvers of Georgia Southern University****.*

Suppose $x^p + y^p = z^p$. If p does not divide x, then by Fermat's Little Theorem, $x^{p-1} \equiv 1 \pmod{p}$ and $x^p \equiv x \pmod{p}$. If $p \mid x$, then $x^p \equiv 0 \equiv x \pmod{p}$, so in either case, $x^p \equiv x \pmod{p}$. Similarly, $y^p \equiv y$ and $z^p \equiv z \pmod{p}$. Thus, if $x^p + y^p = z^p$, then $0 = x^p + y^p - z^p \equiv x + y - z \pmod{p}$, and $p \mid (x + y - z)$.

GRE Practice #11:

Let $f(x) = \sum_{n=1}^{\infty} x^n/n$ for $-1 < x < 1$. Then $f'(x)$ equals
(a) $\frac{1}{1-x}$ (b) $\frac{x}{1-x}$ (c) $\frac{1}{1+x}$ (d) $\frac{x}{1+x}$ (e) 0.
Solution: The answer is (a). If we assume we can differentiate term by term, we get

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$$

Now, hopefully, you recognize this is just the geometric series with ratio x, and the answer is thus (a). For the sake of this exposition, let's assume we didn't notice that. How can we eliminate four answers above?

Notice if we take $x = 0$ then the infinite sum for $f'(x)$ is 1, and this eliminates (b), (d) and (e). So already we are down to just two possibilities. Can we do better? What happens as x approaches 1 from below? Looking at $f(x)$, we see the function is increasing, and the

function is increasing faster as x nears 1 than it is near x equals 0. If we look at the two remaining candidates, (a) and (c), we see $f'(1-\epsilon) > f'(0)$ for (a) but $f'(1-\epsilon) < f'(0)$ for (c). Thus the answer is (a).

The key idea here is to try to look at some special values; we do not need to know exactly what is going on, we just need to have a good enough sense to eliminate some options.

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