# PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2023 

STEVEN J. MILLER (EDITOR)

## 1. Problems: Fall 2023

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk $\left({ }^{*}\right)$ preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $\$ 100$ ). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). The winning school from the Fall problem set is Samford University.


Figure 1. Pizza motivation; can you name the theorem that's represented here?

First Note: After the Spring 2023 issue went to press, we received some additional correct solutions and want to acknowledge the authors: \#1388 was solved by Nago Asahi, Aakash Gurung, and Xuan Pham from Juniata College and Ivan Hadinata (who also solved \#1392 and \#1393) from Gadjah Mada University, Yogyakarta, Indonesia, and by Carl Libis of Columbia Southern University; \#1392 was solved by $\pi$ rates of change, Newark Academy.

Second Note: The Problem Section of this issue is dedicated to the memory of Louis Pfohl. The following note is from one of his children (on a personal note, when I search for his name in my email files, I smile as the display begins "1-50 of many"): He played tennis, basketball, cooked, sketched, and played piano by ear. He worked for Bell Aerospace and Kistler Instruments as an aerospace engineer. He worked on the Reagan Star Wars Program designing anti-ballistic missiles. He was a loving husband to my mom, a caring dad to three children and "Papa Lou" to 6 grandchildren. He loved mathematics and passed that love down to me and his granddaughter (my niece) Ali. In the last years of his life, his favorite pastime was emailing math puzzles from your website page back and forth to Ali and me. We named our trio "The Nerd Club". We would email, call and text - collaborating on the real hard problems until we would come up with a solution/s. My dad loved your website. My last and fondest memories of my dad were working on problems together and calling each other at all hours of the day (and sometimes night) to share an "aha" moment.
\#1400: Proposed by Brian Ha and Steven Miller, Williams College. Assume $f$ satisfies a modified mean value property: for any $z \in \mathbb{C}$ there is some radius $r=r(z)$ such that $f(z)$ is the average of $f(\zeta)$ as $\zeta$ runs over the boundary of the circle of radius $r(z)$. Explicitly: there is a circle $\gamma(z)$ of radius $r(z)$ centered at $z$ such that

$$
\frac{1}{2 \pi r(z)} \oint_{\gamma(z)} f(\zeta) d \zeta=f(z)
$$

Prove or disprove: $f$ is holomorphic (i.e., complex differentiable).
\#1401: Proposed by Hongwei Chen (Christopher Newport University). Let $m$ be a positive integer. For $|q|<1$, prove

$$
\sum_{n=0}^{\infty} \cos \left(\frac{(2 n+1) \pi}{m}\right) q^{n(n+1) / 2}=\cos (\pi / m) \prod_{n=1}^{\infty}\left(1+2 \cos (2 \pi / m) q^{n}+q^{2 n}\right)\left(1-q^{n}\right)
$$

Use this result to deduce the recent Monthly problem 12289:

$$
\sum_{n=0}^{\infty} 2 \cos \left(\frac{(2 n+1) \pi}{3}\right) q^{n(n+1) / 2}=\prod_{n=1}^{\infty}\left(1-q^{6 n-1}\right)\left(1-q^{6 n-5}\right)\left(1-q^{n}\right)
$$

Motivation. One of the most beautiful identity involving two parameters is perhaps Jacobi's triple product identity, which is given by

$$
\begin{equation*}
\sum_{n=-\infty}^{\infty} z^{n} q^{n^{2}}=\prod_{n=1}^{\infty}\left(1+z q^{2 n-1}\right)\left(1+z^{-1} q^{2 n-1}\right)\left(1-q^{2 n}\right) \tag{1}
\end{equation*}
$$

Because its symmetric structure, the simplest proofs of many important theorems follow from this identity. For example, replacing $q$ by $q^{3}$ in (1), then setting $z=-q^{-1}$, we immediately obtain Euler's famous pentagonal number theorem

$$
\sum_{n=-\infty}^{\infty}(-1)^{n} q^{\left(3 n^{2}+n\right) / 2}=\prod_{n=1}^{\infty}\left(1-q^{3 n-1}\right)\left(1-q^{3 n-2}\right)\left(1-q^{3 n}\right)=\prod_{n=1}^{\infty}\left(1-q^{n}\right)
$$

Let $p(n)$ be the number of partitions of $n$. A product representation of the generating function for $p(n)$ has the form:

$$
\sum_{n=0}^{\infty} p(n) q^{n}=\prod_{n=1}^{\infty} \frac{1}{1-q^{n}}
$$

\#1402: Proposed by Steven J. Miller (Williams College) and Elliott Weinstein (Centers for Medicare $\mathcal{E}$ Medicaid Services). You buy a quick pick-k lottery ticket, that is, for fixed $k \geq 1$, a lottery ticket with $k$ numbers $m_{1}, m_{2}, \ldots, m_{k}$ on it generated randomly without replacement from the numbers $1,2, \ldots, N$ for some $N$, but you have no other information as to what $N$ is. What is your best guess for $N$, and thus what is your best guess for the probability your ticket wins the jackpot, i.e., that all $k$ numbers on your ticket match the $k$ numbers in the lottery drawing?
\#1403: Proposed by Geoffrey Caveney, Veritas Tutor: https://veritastutor. com/ index. html. Mathematician 1 states that a constant $A$ in a certain equation must be the product of an algebraic number and the (natural) logarithm of another algebraic number. Mathematician 2 gives a concrete example of such an equation, in which the constant $A$ has the value $\frac{\pi}{/} 4$. Which of the following statements is true?

- The example provided by Mathematician 2 contradicts the statement by Mathematician 1.
- The example provided by Mathematician 2 is consistent with the statement by Mathematician 1.
- The statement by Mathematician 1 together with the example provided by Mathematician 2 imply that the numbers $\pi$ and $e$ (the base of the natural logarithm) are not algebraically independent.
- The statement by Mathematician 1 together with the example provided by Mathematician 2 imply that the number $e^{\pi}$ is an algebraic number.
\#1404: Proposed by Leo Hong, University of North Carolina at Charlotte. Define a great number as a 10 digit number where each digit from 0 to 9 inclusive is used once and only once. (1) Does there exist a great number $G$ whose double is also great? (2) How many great numbers $G$ are there whose double is also great?
\#1405: Proposed by Steven J. Miller (Williams College), Rajaram Venkataramani and Anand Mohanram. Let $p, p+2$ be odd twin primes at least 5; for example 5 and 7, 71 and 73 , or $71,733,689$ and $71,733,691$. Multiply the two primes, and sum the digits. If the sum is not a one digit number, sum the digits again, and keep doing this until a one digit number arises. For example, for our three pairs we get 5 and 7 yields 35 , so the digit sum is 8 , while

71 and 73 has a product of 5,183 whose digit sum is 17 whose digit sum is 8 , and the last pair's product is $5,145,722,281,016,099$ is 62 which then gives a digit sum of 8 . Is this a coincidence or will we always end with an 8 ?

### 1.1. Consequences of Conditions. GRE Practice \#12:

The following is modified from a BC Calculus practice test. Let $f(x)$ be a twice continuously differentiable function such that $f(x)=f(4-x)$. Which of the following conditions must be true: (I) $\int_{0}^{4} f(x) d x \geq 0$. (II) $f^{\prime}(0)=f^{\prime}(4)=1$. (III) $f^{\prime}(2)=0$.
(a) (II) only
(b) (III) only
(c) (I) and (III) only
(d) (II) and (III) only
(e) (I), (II) and (III)

## 2. Solutions

\#1369: Proposed by Steven J. Miller and Chenyang Sun (Williams College). Consider an $n \times n$ chessboard. The previous problem is greatly simplified if instead of queens we place rooks. Determine an optimal placement of $n$ rooks to maximize the number of safe squares.

Solution by Sarah Westmoreland, Kendall Bearden, Chad Awtrey and Frank Patane, Samford University.

Consider an $n \times n$ chessboard with $n$ rooks placed on it. Call a space safe if a pawn can be placed there with no possibility of being captured by any rook in a single move. Among all possible placements of rooks, let $s(n)$ denote the maximum number of safe spaces. Let $m=\lfloor\sqrt{n}\rfloor$; that is, the greatest integer less than or equal to $\sqrt{n}$. We will show:

$$
s(n)= \begin{cases}(n-m)^{2} & \text { if } n=m^{2} \\ (n-m)(n-m-1) & \text { if } m^{2}<n \leq m(m+1), \\ (n-m-1)^{2} & \text { if } n>m(m+1)\end{cases}
$$

In addition, we will describe a corresponding placement of rooks that produces $s(n)$ safe spaces.

For an arbitrary placement of rooks, let $r_{i}$ be the number of rooks in row $i$ of the board, and let $r$ be the maximum among all $r_{i}$ values. Similarly, let $c_{i}$ be the number of rooks in column $i$, and let $c$ be the maximum among all $c_{i}$ values. Since the number of safe spaces is preserved by any permutation of the rows and any permutation of the columns, we may therefore assume that all rows with $r_{i}>0$ are adjacent to each other and all columns with $c_{i}>0$ are adjacent to each other. Further, we may assume that $r_{1}=r$ and $c_{1}=c$. It follows that $s(n)$ is the maximum value of $(n-r)(n-c)$ among all possible placements; this represents the area of the rectangular portion of the board consisting of the bottom right $n-r$ rows and $n-c$ columns. By an optimization argument from first-semester calculus, we know that among all rectangles of a given perimeter, the one with the maximum area is a square. Thus any placement of rooks that has $s(n)$ as the number of safe spaces must be one such that $r$ and $c$ are as close to being equal as possible.

If $n$ is a perfect square (so that $n=m^{2}$ ), then to maximize the number of safe spaces it must be the case that $r=c=m$. In other words, we place rooks in the first $m$ columns
and $m$ rows; that is, $c_{i}=m$ for all $1 \leq i \leq m$ and $c_{i}=0$ for all $i>m$. In this case, $s(n)=(n-m)^{2}$. If $n$ is not a perfect square and $n \leq m(m+1)$, we can maximize the number of safe spaces by placing rooks in the first $m$ columns and the first $m+1$ rows, so that $c_{i}=m+1$ for $1 \leq i \leq n-m^{2}, c_{i}=m$ for $n-m^{2}<i \leq m$, and $c_{i}=0$ for $i>m$. In this case, we have $s(n)=(n-m)(n-m-1)$. For the final case, when $n>m(m+1)$, in order to maximize the number of safe spaces we can place rooks in the first $m+1$ columns and the first $m+1$ rows. Specifically, we have $c_{i}=m+1$ for $1 \leq i \leq m, c_{m+1}=n-m(m+1)$, and $c_{i}=0$ for $i>m+1$. In this case, we have $s(n)=(n-m-1)^{2}$.
\#1372: Proposed by Steven J. Miller and Chenyang Sun (Williams College).
The following is a standard problem, with a generalization that has applications in understanding some methods in Operations Research. Consider a positive integer $N \geq 100$. (a) We want to divide $N$ into positive integer pieces $a_{1}, \ldots, a_{n}$ such that the product $a_{1} \cdots a_{n}$ is as large as possible. How do we do this? (b) What if we just require the pieces to be positive numbers: how do we do it in that case?

Solution by Sarah Westmoreland, Bailey Holland, Frank Patane and Chad Awtrey, Samford University.
(a) Let $N=a_{1}+a_{2}+\cdots+a_{n}$ be a partition of $N$ into positive integer parts. We seek to maximize the product $P=a_{1} a_{2} \cdots a_{n}$ over all possible partitions. Now if $a_{1}=1$ we have $N=1+a_{2}+\cdots+a_{n}$ and $P=1 \cdot a_{2} a_{3} \cdots a_{n}<\left(1+a_{2}\right) a_{3} \cdots a_{n}$. That is, we can make the product larger if we view $N$ as having $n-1$ parts and add $a_{1}=1$ to another part. Hence we can take $a_{i}>1$ for all $i=1, \ldots, n$. We also see that if $a_{i}>4$ then we write $a_{i}=\left(a_{i}-3\right)+3$ and $a_{i}<3\left(a_{i}-3\right)$ implies the product of parts would be larger if we split $a_{i}$ into the two parts $a_{i}-3$ and 3. Hence we assume $a_{i} \in\{2,3,4\}$ for all $i$. Now $4=2+2=2 \cdot 2$ and so any part of size 4 can be split into two parts of size two and the product $P$ will remain unchanged. Thus we take $a_{i} \in\{2,3\}$ for all $i$. Lastly, we see $6=2+2+2=3+3$ and since $2 \cdot 2 \cdot 2<3 \cdot 3$ the product is maximized if we take as many parts equal to 3 as possible. So the way to maximize $P$ is if $N \equiv 0 \bmod 3$ then take all parts equal to 3 , if $N \equiv 1 \bmod 3$ then take two parts equal to 2 and the rest equal to 3 , and lastly if $N \equiv 2 \bmod 3$ then take 1 part equal to 2 and the rest equal to 3 .
(b) Let $x_{1}, \ldots, x_{n}$ be nonnegative real numbers that sum to $N$. We want to maximize the function $f\left(x_{1}, \ldots, x_{n}\right)=x_{1} \cdots x_{n}$ subject to the constraint function $g\left(x_{1}, \ldots, x_{n}\right)=N$ where $g\left(x_{1}, \ldots, x_{n}\right)=x_{1}+\cdots+x_{n}$.

Using the method of Lagrange multipliers, this amounts to solving the system of equations: $\partial f / \partial x_{i}=\lambda \partial g / \partial x_{i}$ and $g=N$. In particular, we want to solve:

$$
\begin{aligned}
\lambda & =x_{2} \cdots x_{n} \\
\lambda & =x_{1} x_{3} \cdots x_{n} \\
\vdots & \\
\lambda & =x_{1} \cdots x_{n-1} \\
N & =x_{1}+\cdots+x_{n}
\end{aligned}
$$

If $\lambda=0$, then at least two $x_{i}$ values are 0 ; such solutions result in $f=0$. If $\lambda \neq 0$, notice that the first $n$ equations show that $x_{i}=x_{j}$ for all $1 \leq i, j \leq n$. Combining this with the final equation shows that the unique solution occurs when $x_{i}=N / n$; i.e., all inputs are equal. In this case, we clearly achieve a max function value of $(N / n)^{n}$.

In order to determine which value of $n$ maximizes $f$, it suffices to maximize the singlevariable function $h(z)=(N / z)^{z}$. Since we are ultimately interested in maximizing $h$ among all positive integer values of $z$, we restrict the domain to $z \geq 1$. We can further restrict the domain to $z \leq N$, since $h(N)=1$ and $h(z)<1$ for $z>N$. We see that $h(z)$ is continuous and defined on a compact set and therefore does achieve a maximum value.

Using logarithmic differentiation, it follows that $h^{\prime}(z)=h(z) \cdot(\log (N / z)-1)$. Thus $h^{\prime}(z)=0$ when $1=\log (N / z)$; that is, when $z=N / e$. This critical number produces a maximum since $h^{\prime}(z)<0$ for $z>N / e$ and $h^{\prime}(z)>0$ for $z<N / e$.

Therefore, among all positive integer values of $z$, we see that the maximum value for $h(z)$ occurs either when $z=\lfloor N / e\rfloor$ (the greatest integer less than or equal to $N / e$ ) or when $z=\lceil N / e\rceil$ (the least integer greater than or equal to $N / e$ ); whichever produces the greater function value.
\#1394: Proposed by Steven Miller, Williams College. When we dropped my son off at Camp Winadu we received a blue and white half moon (those are his camp's colors). A half moon is a delicious frosted cookie, with frosting of one color on half of the circle and another color on the other. Splitting it in two is easy to do so that everyone gets an equal amount of the two colors. What is a good way to split it into three equal parts where each has the same amount of each color? Describe exactly where you make the cuts, assume all you can do is cut in any straight line. (Note of course it is trivial if you do not care about the amount of each color; choose 3 equi-spaced points on the perimeter and cut from the center to these.)

## Solution by Daniel Podzunas, Western New England University.

We define the color-dividing line (CDL) as the line separating the two different colored halves of the cookie. As illustrated below, we will make straight cuts from the points a distance $k$ away from the center along the CDL to the end points on the diameter perpendicular to the CDL. Let $r$ be the radius of the cookie, we solve for $k$ in the equation $\frac{r^{2} \pi}{3}=2 k r$ and obtain that $k=\frac{r \pi}{6}$. Noting that $k$ is one twelfth of the cookie circumference, one could measure the circumference with a rope and fold it into 12 equal parts to obtain the length of $k$ in real life.


Comparing to a more intuitive 2-horizontal-cut attempt, we would like to address that our solution created an arc-free middle piece and along with a pair of mirrored top and bottom pieces. This allowed us to work with simpler geometry for a solution that could be easily used to determine cut placement.

Note from the editor: As remarked above, one can also solve this by two horizontal cuts; in both methods it is easy to cut in approximately the right spot or right angle, but in practice it would be difficult to hit the exact value. For example, in the solution above one would have to mark off $\pi / 6$ units, and thus somehow 'construct' $\pi$.
\#1395: Proposed by Hongwei Chen, Christopher Newport University. Show that the Fourier sine series of $\ln (\tan x)$ on $(0, \pi / 2)$ is given by

$$
\begin{equation*}
\ln (\tan x)=-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{h_{n}}{n} \sin (4 n x) \tag{1}
\end{equation*}
$$

where

$$
h_{n}=1+\frac{1}{3}+\cdots+\frac{1}{2 n-1} .
$$

Motivation: It is well-known that $\{\cos (2 n x)\}_{n=0}^{\infty}$ forms an orthogonal basis of $L^{2}(0, \pi / 2)$. In particular, we have

$$
\begin{aligned}
& -\ln (\sin x)=\ln 2+\sum_{k=1}^{\infty} \frac{1}{k} \cos (2 k x) \\
& -\ln (\cos x)=\ln 2+\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k} \cos (2 k x) .
\end{aligned}
$$

This yields the Fourier cosine series of $\ln (\tan x)$ :

$$
\begin{equation*}
-\ln (\tan x)=2 \sum_{k=0}^{\infty} \frac{1}{2 k+1} \cos (2(2 k+1) x), \quad x \in(0, \pi / 2) \tag{2}
\end{equation*}
$$

It is natural to ask for its corresponding Fourier sine series. Since

$$
h_{n}=H_{2 n}-\frac{1}{2} H_{n},
$$

where $H_{n}$ is the $n$th harmonic number, our formula enables us to find some exact values of Euler sums. For example, applying Parseval's identity, together with

$$
\int_{0}^{\pi / 2} \ln ^{2}(\tan x) d x=\frac{1}{8} \pi^{3}
$$

we recover the identity

$$
\sum_{n=1}^{\infty} \frac{1}{n^{2}} h_{n}^{2}=\frac{1}{32} \pi^{4}
$$

Solution by Seán Stewart, King Abdullah University of Science and Technology.
We begin by starting with the well-known Maclaurin series expansions for $\log (1+z)$ and $\log (1-z)$, namely

$$
\log (1+z)=\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{n} \quad \text { and } \quad \log (1-z)=-\sum_{n=1}^{\infty} \frac{z^{n}}{n}
$$

Here $z \in \mathbb{C}$, $\log$ is the principal branch of the logarithm defined on $\mathbb{C} \backslash(-\infty, 0]$, while both series expansions are valid for $|z|<1$. Taking their difference one obtains

$$
\log \left(\frac{1+z}{1-z}\right)=2 \sum_{n=1}^{\infty} \frac{1}{2 n-1} z^{2 n-1}
$$

or

$$
\log \left(\frac{1+z}{1-z}\right)=2 z \sum_{n=0}^{\infty} \frac{1}{2 n+1} z^{2 n}
$$

after the index has been shifted by $n \mapsto n+1$. Squaring this result we find

$$
\begin{align*}
\log ^{2}\left(\frac{1+z}{1-z}\right) & =2 z \sum_{n=0}^{\infty} \frac{1}{2 n+1} z^{2 n} \cdot 2 z \sum_{m=0}^{\infty} \frac{1}{2 m+1} z^{2 m} \\
& =4 z^{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(2 k+1)(2(n-k)+1)} z^{2 n} \tag{2.1}
\end{align*}
$$

where in the second line we have taken the Cauchy product. From the partial fraction decomposition of

$$
\frac{1}{(2 k+1)(2 n-2 k+1)}=\frac{1}{2(n+1)(2 k+1)}-\frac{1}{2(n+1)(2 k-2 n-1)},
$$

the finite sum appearing in (2.1) can be written as

$$
\begin{aligned}
\sum_{k=0}^{n} \frac{1}{(2 k+1)(2(n-k)+1)} & =\frac{1}{2(n+1)} \sum_{k=0}^{n} \frac{1}{2 k+1}-\frac{1}{2(n+1)} \sum_{k=0}^{n} \frac{1}{2 k-2 n-1} \\
& =\frac{1}{2(n+1)} \sum_{k=0}^{n} \frac{1}{2 k+1}+\frac{1}{2(n+1)} \sum_{k=0}^{n} \frac{1}{2 k+1} \\
& =\frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{2 k+1}=\sum_{k=0}^{n} \frac{h_{n+1}}{n+1} .
\end{aligned}
$$

Note in the second line, a reindexing of $k \mapsto n-k$ was made in the sum on the right. Thus (2.1) becomes

$$
\log ^{2}\left(\frac{1+z}{1-z}\right)=4 z^{2} \sum_{n=0}^{\infty} \frac{h_{n+1}}{n+1} z^{2 n}=4 \sum_{n=0}^{\infty} \frac{h_{n}}{n} z^{2 n}, \quad|z|<1,
$$

after a reindexing of $n \mapsto n-1$ has been made.
Setting $z=e^{2 i x}$ where $i$ is the imaginary unit and $x \in(0, \pi / 2)$, since

$$
\begin{aligned}
\log ^{2}\left(\frac{1+z}{1-z}\right) & =\log ^{2}\left(\frac{1+e^{2 i x}}{1-e^{2 i x}}\right)=\log ^{2}(i \cot x) \\
& =\left(\log (\cot x)+\frac{i \pi}{2}\right)^{2} \\
& =\log ^{2}(\cot x)+i \pi \log (\cot x)-\frac{\pi^{2}}{4}
\end{aligned}
$$

we see that

$$
\log ^{2}(\cot x)+i \pi \log (\cot x)-\frac{\pi^{2}}{4}=4 \sum_{n=1}^{\infty} \frac{h_{n}}{n}(\cos (4 n x)+i \sin (4 n x))
$$

Equating the imaginary part yields

$$
\pi \log (\cot x)=4 \sum_{n=1}^{\infty} \frac{h_{n}}{n} \sin (4 n x)
$$

or

$$
\log (\tan x)=-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{h_{n}}{n} \sin (4 n x), \quad x \in(0, \pi / 2)
$$

as required to show.
As a bonus, equating the real part yields

$$
\log ^{2}(\cot x)-\frac{\pi^{2}}{4}=4 \sum_{n=1}^{\infty} \frac{h_{n}}{n} \cos (4 n x)
$$

or

$$
\log ^{2}(\tan x)=\frac{\pi^{2}}{4}+4 \sum_{n=1}^{\infty} \frac{h_{n}}{n} \cos (4 n x), \quad x \in(0, \pi / 2)
$$

which is the Fourier cosine series of $\log ^{2}(\tan x)$. From this result we see that the result given for the value of the integral appearing in the Motivation section immediately follows.

## GRE Practice \#12:

The following is modified from a BC Calculus practice test. Let $f(x)$ be a twice continuously differentiable function such that $f(x)=f(4-x)$. Which of the following conditions must be true: (I) $\int_{0}^{4} f(x) d x \geq 0$. (II) $f^{\prime}(0)=f^{\prime}(4)=1$. (III) $f^{\prime}(2)=0$.
(a) (II) only
(b) (III) only
(c) (I) and (III) only
(d) (II) and (III) only
(e) $(\mathrm{I}),(\mathrm{II})$ and (III)

Solution: We can start trying to solve the problem. If we differentiate $f(x)=f(4-x)$ we get $f^{\prime}(x)=-f^{\prime}(4-x)$; an excellent choice of $x$ is 2 , as then both sides evaluate $f$ at the same point, giving $f^{\prime}(2)=-f^{\prime}(2)$. Thus $f^{\prime}(2)=0$, and we see (III) is true, which eliminates (a). If we take $x=0$ we get $f^{\prime}(0)=-f^{\prime}(4)$, so (II) is false as the only way they can be equal is if they are both zero. Thus it cannot be (d) or (e) and we need to determine if (I) holds. To determine if (I) must be true we note that if $f(x)$ satisfies the sole condition, so too does $g(x):=-f(x)$, and this has the effect of multiplying the integral by -1 . Thus so long as the integral is not zero we have a contradiction; clearly the integral need not vanish as taking $f(x)=c$ shows for any fixed $c \neq 0$.

Our last argument suggests a faster way to solve the problem: rather than trying to prove which of the three statements are correct (which is not too bad in this case), we can try to find special functions and see. Two good choices that have $f(x)=f(4-x)$ are $f_{1}(x)=c$ and $f_{2}(x)=c x(4-x)$ for a fixed constant $c$. We immediately see that (I) and (II) are not always true as they fail for certain choices of $c$. The derivatives are $f_{1}^{\prime}(x)=0$ and $f_{2}^{\prime}(x)=\left(4 c x-c x^{2}\right)^{\prime}=c(4-2 x)$, both of which vanish at $x=2$, and thus (III) may be true.

Finally, note when taking the derivative of $f_{2}(x)$ it was easier to expand and then differentiate, rather than use the product rule. This is often the case - it is worth taking a moment to simplify an expression before doing a calculation.

Email address: sjm1@williams.edu
Professor of Mathematics, Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

