PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2023

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2024

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to \$100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). The winning school for this issue is **Georgia Southern University**.

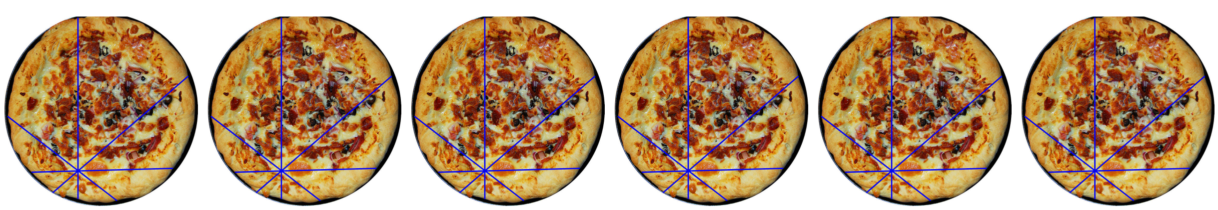


Figure 1. Pizza motivation; can you name the theorem that's represented here?

Date: April 1, 2024.

Each year a distinguished mathematician gives the J. Sutherland Frame Pi Mu Epsilon Lecture at MathFest. In 2019 that speaker was Alice Silverberg, Distinguished Professor at the University of California, Irvine; her talk is available at , and Problem #1366 is inspired by her lecture. For 2020 the speaker was to be Florian Luca from the University of the Witwatersrand and we were to have a problem based on his talk, but MathFest canceled due to the covid response; we hope to share such inspired problems again in the future.

#1406: *Proposed by Faird Jokar, Shahid Rajaee Teacher Training University, Tehran, Iran.* We denote $\overline{a_k a_{k-1} \ldots a_1}$ as the number $a_k \times 10^{k-1} + a_{k-1} \times 10^{k-2} + \cdots + a_2 \times 10 + a_1$ for positive integer k. If $a = \overline{a_k a_{k-1} \ldots a_1}$ and $b = \overline{b_{k'} b_{k'-1} \ldots b_1}$ for positive integers k and k', then we define $\widehat{ab}$ as the number $\overline{a_k a_{k-1} \ldots a_1 b_{k'} b_{k'-1} \ldots b_1}$. It is easy to see that there are infinitely many square numbers a and b such that $\widehat{ab}$ is square. For instance, 4 and 9 are two square numbers. By putting 4 and 9 together, we construct 49 which is another square number. More generally, for every positive integer i, 49×10^{2i} is a square number which is constructed by putting 4 and 9×10^{2i} together, in which both 4 and 9×10^{2i} are square numbers. Prove or disprove: there are infinitely many square numbers a and b such that $\gcd(a, 10) = \gcd(b, 10) = 1$, and also $\widehat{ab}$ is square.[1]

#1407: *Proposed by Joe Santmyer, US Federal Government (retired).* Finding zeros of a function and their properties occupies a large literature in mathematics. Many solved and unsolved problems deal with zeros of a function. Notable statements, such as the Fundamental Theorem of Algebra and the Riemann Hypothesis, are center stage but many lesser known results are scattered in the literature.
The problem here was motivated by an exercise on page 155 in Stein and Shakarchi's *Princeton Lectures in Analysis II: Complex Analysis*, which is to prove that the entire function $f(z) = e^z - z$ has an infinite number of zeros. If an analytic function that is not identically zero has an infinite number of zeros then they are countable. Let $\{a_n\}$ be the sequence of zeros of f. What else can be said about the zeros? Prove one can at least say the following.

a. $\sum\limits_{n=1}^{\infty} \frac{1}{a_n(1-a_n)} = 1$

b. $\sum\limits_{n=1}^{\infty} \frac{1}{a_n(2\pi i m-a_n)} = 0$ where m is a nonzero integer

c. $\sum\limits_{n=1}^{\infty} \frac{1}{a_n^2} = -1$

d. $\sum\limits_{n=1}^{\infty} \frac{1}{a_n^3} = -\frac{1}{2}$.

#1408: *Proposed by Ron Evans.* There is a long, rich history of trying to find which equations with integer coefficients have integer solutions (and if there are solutions, determining them). The famous Pell equation is $x^2 - dy^2 = 1$ for d-square free. More generally,

[1]A couple of years ago, the proposer found the following question, which is a weaker version, in a Persian website, which was for Professor Madjid Mirzavaziri: *Are there infinitely many square numbers a and b such that* $\gcd(b, 10) = 1$*, and also $\widehat{ab}$ is square?* This website possessed some interesting questions without their solutions. Unfortunately, this website does not exist now. Professor Mirzavaziri found this question in a note, but does not remember the reference for that.

consider $x^2 - dy^2 = q$ where q is prime. (a) Prove that there are no integer solutions when $(d, q) = (37, 3)$. (b) Find integer solutions for $(d, q) = (79, 5)$ or show there are none.

#1409: *Proposed by Ron Evans and Steven J. Miller (Williams College).* As mentioned in #1409, there is a long, rich history of trying to find which equations with integer coefficients have integer solutions (and if there are solutions, determining them). (a) Find all integer solutions to $3y^2 + 3y + 1 = x^3$. (b) Prove that $12x^3 - 3$ is never a square for $x > 1$. *Hint: Part (a) might be useful.*

#1410: *Proposed by Kenny B. Davenport, St Petersburg, Florida.* The Pell numbers, defined by $P_0 = 0, P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$, are an interesting sequence of numbers with numerous properties; they are one of the simplest generalizations of the Fibonacci recurrence (same initial conditions but now $F_{n+1} = 1F_n + F_{n-1}$), and arise as the denominators in the sequence of the best rational approximations to $\sqrt{2}$. Not surprisingly, they satisfy a large number of interesting relations. Prove

$$2\sum_{k=1}^{n} kP_{k-1} \ = \ nP_{n+1} - (n+1)P_n.$$

Note: depending on the path you take to the proof, you may be able to generate many other additional identities, such as

$$2\sum_{k=1}^{n} k^2 P_{k-1} \ = \ (n^2+1)P_{n+1} - (n^2+2n)P_n - 1.$$

More generally, though you are only asked to prove the identity for the sum of k times P_{k-1}, can you conjecture what the shape of the answer should be for the sum of k^d times P_{k-1}?

The following text is motivation for the next two problems. Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ have attracted the interest of mathematicians dating at least back to Euler; see for example

- Bailey, D., Borwein, J., Girgensohn, R., Experimental evaluation of Euler sums, *Experimental Math.*, **3** (1994), 17–30.
- Borwein, D., Borwein, J., On some intriguing sums involving $\zeta(4)$, *Proc. Amer. Math. Soc.*, **123** (1995), 1191-1198.
- Chen, H., Evaluations of some Euler sums, *J. Integer Seq.*, **9** (2006), Article 06.2.3.
- Chen, H., Excursions in Classical Analysis, Mathematical Association of America, Inc., 2010.
- Gradshteĭn, Ryzhik, I. M., Jeffrey, A. “Table of Integrals, Series and Products”, Academic Press, p. 544, 1994.
- Riordan, J., “Combinatorial Identities”, John Wiley & Sons, Inc., 1968.
- Spieß, J., Some identities involving harmonic numbers, *Math. of Comp.*, Vol. 55, No. 192, Oct. 1990, 839-803.
- Spiegel, M. R., “Mathematical Handbook”, Schaum's Outline Series, McGraw-Hill Book Company, 1968.

These references demonstrate many beautiful formulas contain H_n. There continues to be active research in harmonic numbers and their generalizations as the websites

a. Open Problem
b. Generalized Harmonic Numbers and Combinatorial Sequences
c. Euler Type Sums of Harmonic Numbers
d. Generalized Harmonic Numbers
e. q-Harmonic Numbers
f. Generating Functions of Harmonic Numbers
g. Harmonic Numbers and Integrals

illustrate. The references and websites show the interplay between analysis, combinatorics and number theory that characterizes the study of harmonic numbers. The problems posed here give you an opportunity to experience this interplay. You are ask to prove two formulas. One contains H_n and the other a harmonic like value $h_n = \sum_{k=1}^{n} \frac{1}{2k-1}$.

#1411: *Proposed by Joe Santmyer, US Federal Government (retired)*
With the aide of tables and technology (for example, *Mathematica*) show that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\frac{H_{j+1}}{j+1} \quad = \quad \frac{\pi^2}{12}.$$

#1412: *Proposed by Joe Santmyer, US Federal Government (retired)*
With the aide of tables and technology (for example, *Mathematica*) show that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\frac{h_{j+1}}{j+1} \quad = \quad [\sinh^{-1}(1)]^2.$$

GRE Practice #13: The following is inspired by a practice SAT problem, and the thought process of Cameron Miller in solving it. Let $f(x) = ax^4 + bx^2 + c$, with $f(0) = 1$, $f(1) = -3$ and $f(2) = 9$. What is $f(-1)$?
(a) -2 (b) -3 (c) -4 (d) -5 (e) -6.

2. Solutions

#1375: *Proposed by Charles Audet, Ecole Polytechnique de Montréal.*
A collection of n squares are placed side-by-side. They occupy an area greater than or equal to 1, and the sum of their side lengths does not exceed an integer $k \leq n$. Show that there are k squares whose sum of side lenghts is greater than or equal to 1.
Solution by Drew Middleton, Ariana Allgood, Andrew Hill, Kendall Bearden, Samford Problem Solving Group.

The conclusion is trivially true if the largest piece has a side length greater than one, so without loss of generality, it suffices to study the case when each square side length is less than one. Consider a collection of n squares placed side by side. We reorder these squares by side length, so that $X_1 \geq X_2 \geq \ldots \geq X_n$ where X_i denotes the side length of the ith square. With this notation we are given the following:

$$1 \ \geq \ X_1 \ \geq \ X_2 \ \geq \ \cdots \ \geq \ X_n > 0, \tag{2.1}$$

$$X_1^2 + X_2^2 + \cdots + X_n^2 \ \geq \ 1, \tag{2.2}$$

$$X_1 + X_2 + \cdots + X_n \ \leq \ k \ \leq \ n, \tag{2.3}$$

and we wish to show

$$1 \ \leq \ X_1 + X_2 + \cdots + X_k. \tag{2.4}$$

We note that (2.1) follows from our notation along with $X_1 > 1$ implying (2.4). Hence we assume $1 \geq X_1$. Define the mth partial sum

$$S_m \ = \ \sum_{i=1}^{m} X_i^2.$$

Using (2.1), (2.2), and (2.3) we have the following

$$\begin{aligned}
1 \ &\leq \ S_k + \sum_{i=k+1}^{n} X_i^2 \\
&\leq \ S_k + X_k \sum_{i=k+1}^{n} X_i \\
&\leq \ S_k + X_k(k - \sum_{i=1}^{k} X_i) \\
&= \ S_k + X_k \sum_{i=1}^{k}(1 - X_i) \\
&\leq \ \sum_{i=1}^{k}(X_i^2 + X_k(1 - X_i)) \\
&\leq \ \sum_{i=1}^{k}(X_i^2 + X_i(1 - X_i)) \\
&= \ \sum_{i=1}^{k} X_i.
\end{aligned}$$

Thus (2.4) is shown.

#1399: *Proposed by Zhongxue Lü (Jiangsu Normal University) and Steven J. Miller (Williams College).* This problem is inspired from an observation in 2021 (due to the backlog of problems it is only being published now), where 2021 is formed by writing two consecutive integers one after the other; in other words it is of form $n \cdot 10^k + (n+1)$ where k is the number of digits of n and n has leading digit non-zero and is not all 9's. We call such integers 2-adjacent joined numbers. Note we do not consider 102 or 10000 such numbers (even though the first could be written as $01 * 10^2 + 02$ and the second as $99 * 10^2 + 100$). How many 2-adjacent joined numbers are there less than 10^{100}?

Solution by Daniel Podzunas, Western New England University. Also solved by the Problem Solving Group at Ashland University.

Notice the defined form $n \cdot 10^k + (n+1)$ only results in numbers with an even amount of digits. In particular, $n \cdot 10^k$ and $n+1$ will make up the first k digits and the following k digits of the number, respectively. Fix any k, we want to count how many valid n's there are to form a 2-adjacent joined number.

$k=1$	$k=2$	$k=3$	$k=4$		k
12	1011	100101	10001001	$\cdots$	$[10^{k-1}] \cdot 10^k + ([10^{k-1}] + 1)$
23	1112	101102	10011002	$\cdots$	$\cdots$
34	1213	102103	10021003	$\cdots$	$\cdots$
45	1314	103104	10031004	$\cdots$	$\cdots$
56	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
67	9697	996997	99969997	$\cdots$	$\cdots$
78	9798	997998	99979998	$\cdots$	$\cdots$
89	9899	998999	99989999	$\cdots$	$[10^k - 2] \cdot 10^k + ([10^k - 2] + 1)$

As we observe in the table that a valid n ranges from 10^{k-1} to $10^k - 2$, we could form

$$(10^k - 2) - (10^{k-1}) + 1 = 10 \cdot 10^{k-1} - 10^{k-1} - 1 = 9 \cdot 10^{k-1} - 1$$

many 2-adjacent joined numbers for any given k. Since the largest number we are looking at is 10^{100}, we are adding up to $k = 50$. Thus, there are

$$\sum_{k=1}^{50} 9 \cdot 10^{k-1} - 1 = 99999999999999999999999999999999999999999999999999 - 50$$
$$= 99999999999999999999999999999999999999999999999949$$

2-adjacent joined numbers.

Note: the following approach was sent as an alternative to the counting by cases. The n's of any length can be considered as all having 50 digits with possibly many leading zeros. Then out of the 10^{50} choices for 50 digit numbers we must exclude the one that is all 0's, and the 50 that have leading zeros followed by all 9's. Hence there are $10^{50} - 51$ choices for n.

#1401: *Proposed by Hongwei Chen (Christopher Newport University).* Let m be a positive integer. For $|q| < 1$, prove

$$\sum_{n=0}^{\infty} \cos\left(\frac{(2n+1)\pi}{m}\right) q^{n(n+1)/2} = \cos(\pi/m) \prod_{n=1}^{\infty} (1 + 2\cos(2\pi/m) q^n + q^{2n})(1 - q^n). \quad (2.5)$$

Use this result to deduce the recent Monthly problem 12289:

$$\sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{n(n+1)/2} = \prod_{n=1}^{\infty}(1-q^{6n-1})(1-q^{6n-5})(1-q^n). \tag{2.6}$$

Solution by the Missouri State University Problem Solving Group.

Given Jacobi's triplet product identity

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} \;=\; \prod_{n=1}^{\infty}(1+zq^{2n-1})(1+z^{-1}q^{2n-1})(1-q^{2n}),$$

we set $z = qe^{2\tau i}$ for some τ with $i^2 = -1$, yielding

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} q^{n^2+n}e^{2n\tau i} \;&=\; \prod_{n=1}^{\infty}(1+q^{2n}e^{2\tau i})(1+q^{2n-2}e^{-2\tau i})(1-q^{2n}) \\
&=\; (1+e^{-2\tau i})\prod_{n=1}^{\infty}(1+q^{2n}e^{2\tau i})(1+q^{2n}e^{-2\tau i})(1-q^{2n}) \\
&=\; (1+e^{-2\tau i})\prod_{n=1}^{\infty}(1+q^{2n}(e^{2\tau i}+e^{-2\tau i})+q^{4n})(1-q^{2n}) \\
&=\; (1+e^{-2\tau i})\prod_{n=1}^{\infty}(1+2\cos(2\tau)q^{2n}+q^{4n})(1-q^{2n}). \qquad (2.7)
\end{aligned}$$

Multiplying both sides of (2.7) by $e^{\tau i}$, we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} q^{n^2+n}e^{(2n+1)\tau i} \;&=\; (e^{\tau i}+e^{-\tau i})\prod_{n=1}^{\infty}(1+2\cos(2\tau)q^{2n}+q^{4n})(1-q^{2n}) \\
&=\; 2\cos\tau\prod_{n=1}^{\infty}(1+2\cos(2\tau)q^{2n}+q^{4n})(1-q^{2n}). \qquad (2.8)
\end{aligned}$$

On the left-hand side of (2.8), we notice that the terms with $n = -k$ and $n = k-1$ are conjugate for all integer k. Using Euler's identity, we have

$$\sum_{n=-\infty}^{\infty} q^{n^2+n}e^{(2n+1)\tau i} \;=\; \sum_{n=1}^{\infty} 2q^{n^2+n}\cos\left((2n+1)\tau\right). \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$\sum_{n=1}^{\infty} q^{n^2+n}\cos\left((2n+1)\tau\right) \;=\; \cos\tau\prod_{n=1}^{\infty}(1+2\cos(2\tau)q^{2n}+q^{4n})(1-q^{2n}). \tag{2.10}$$

Finally, in (2.10), replacing q by $\sqrt{q}$ and replacing τ by π/m, we obtain (2.5).

Next, we will derive (2.6) from (2.5). Setting $m = 3$, (2.5) becomes

$$\sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{n(n+1)/2} = \prod_{n=1}^{\infty}(1-q^n+q^{2n})(1-q^n)$$
$$= \prod_{n=1}^{\infty} \frac{1+q^{3n}}{1+q^n}(1-q^n) = \frac{\prod_{n=1}^{\infty}(1+q^{3n})}{\prod_{n=1}^{\infty}(1+q^n)} \prod_{n=1}^{\infty}(1-q^{2n-1}) \prod_{n=1}^{\infty}(1-q^{2n})$$
$$= \prod_{n=1}^{\infty}(1+q^{3n}) \prod_{n=1}^{\infty}(1-q^{2n-1}) \prod_{n=1}^{\infty}(1-q^n)$$
$$= \prod_{n=1}^{\infty}(1+q^{3n}) \prod_{n=1}^{\infty}(1-q^{6n-3}) \prod_{n=1}^{\infty}(1-q^{6n-1})(1-q^{6n-5}) \prod_{n=1}^{\infty}(1-q^n). \tag{2.11}$$

However,

$$\prod_{n=1}^{\infty}(1+q^{3n}) \prod_{n=1}^{\infty}(1-q^{6n-3}) = \prod_{n=1}^{\infty}(1+q^{3n}) \prod_{n=1}^{\infty}(1-q^{3(2n-1)})$$
$$= \prod_{n=1}^{\infty}(1+q^{3n}) \prod_{n=1}^{\infty} \frac{(1-q^{3(2n-1)})(1-q^{3(2n)})}{1-q^{3(2n)}}$$
$$= \prod_{n=1}^{\infty}(1+q^{3n}) \frac{\prod_{n=1}^{\infty}(1-q^{3n})}{\prod_{n=1}^{\infty}(1-q^{6n})} = 1. \tag{2.12}$$

Substituting the result in (2.12) to (2.11), we get

$$\sum_{n=0}^{\infty} 2\cos\left(\frac{(2n+1)\pi}{3}\right) q^{n(n+1)/2} = \prod_{n=1}^{\infty}(1-q^{6n-1})(1-q^{6n-5})(1-q^n).$$

#1404: *Proposed by Leo Hong, University of North Carolina at Charlotte.* Define a *great number* as a 10 digit number where each digit from 0 to 9 inclusive is used once and only once. (1) Does there exist a great number G whose double is also great? (2) How many great numbers G are there whose double is also great?

Solution by Jesús Sistos (student) and the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

There exist many such numbers. For example, $G = 4,560,172,893$ is great and $2G = 9,120,345,786$ is also great. In total there are $184,320$ great numbers whose double is also great. For brevity, we will call such numbers *doubly great*.

We analyze this doubling digit by digit. Doubling is localized in the sense that carrying does not have any effect on digits that are not immediately to the left of the digit generating the carryover. (Notice that this is not the case when multiplying by a larger integer, such as 3: in the product $35 \cdot 3 = 105$, the carryover from tripling the units digit 5 extends two digits to the left.) When doubling a number, a digit generates a carryover if and only if the digit is an element of $C = \{5, 6, 7, 8, 9\}$.

In a great number, we say that a digit is a *giver* if it is an element of C. Moreover, we say that a digit is a *receiver* if it is to the left of a giver. Because all great numbers have exactly 10 digits, the leftmost digit of a doubly great number cannot be a giver, since that would generate an additional (eleventh) digit. By definition, the rightmost digit cannot be a receiver, since it is not to the left of any digit.

Notice that after doubling, a 0 digit becomes either a 0 or a 1, depending on whether the original 0 digit is a receiver; the same is true for the digit 5. Indeed, the pairs $\{0,5\}$, $\{1,6\}$, $\{2,7\}$, $\{3,8\}$, and $\{4,9\}$ generate elements of a common pair of digits: respectively, $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{6,7\}$, and $\{8,9\}$. Since all ten digits must arise from the doubling of a doubly great number, exactly one number in each pair is a receiver. For example, the pattern 3586 cannot appear in any doubly great number because both 3 and 8 are receivers, and hence would generate the same digit, 7, when doubled.

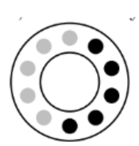
For convenience, instead of counting individual doubly great numbers, we will count doubly great cycles; that is, cyclic arrangements of 10-digit integers. Formally, these are equivalence classes of doubly great numbers under the relation of cyclic shifts. We can cut a cycle by choosing an initial digit, and then reading the digits clockwise, starting at the cut. For a doubly great number, we cannot cut a cycle at 0, or at any digit in C, since the leftmost digit of a doubly great number can be neither 0 nor a giver. Therefore, each doubly great cycle corresponds to exactly four doubly great numbers.

For example, our original doubly great number $G = 4,560,172,893$ can be turned into a doubly great cycle by adjoining its leftmost and rightmost digits in a cycle, which we denote [4560172893]. We may then cut this cycle at any of the digits in $\{1,2,3,4\}$ to obtain three additional doubly great numbers: in this case, $1,728,934,560$; $2,893,456,017$; and $3,456,017,289$.

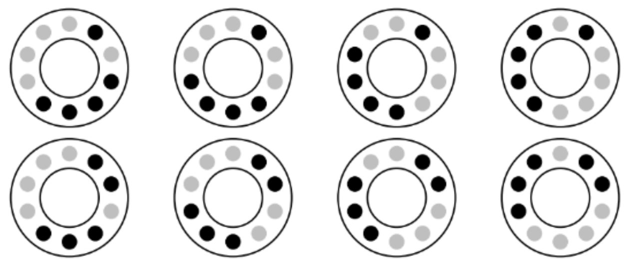
To count the number of doubly great cycles, we first say that an ordered list of contiguous giver digits in a cycle is a *block* if it is of maximal size. For example, in the cycle [0123456789], the digits 678 are givers, but they do not form a block because they are contained in a larger set of contiguous givers, namely 56789, which is the only block in [0123456789]. In the cycle [4560172893], there are three blocks; namely, 56, 7, and 89.

The distribution of the blocks will determine the behavior of the intersection of the set of givers and the set of receivers. Notice that in a block, all digits that are not the rightmost digit in the block are also receivers. By associating each block with its rightmost digit, we observe that the number of blocks is the same as the number of givers that are not also receivers. We break up the possibilities by looking at the number of blocks in each cycle; illustrations are given for each case.

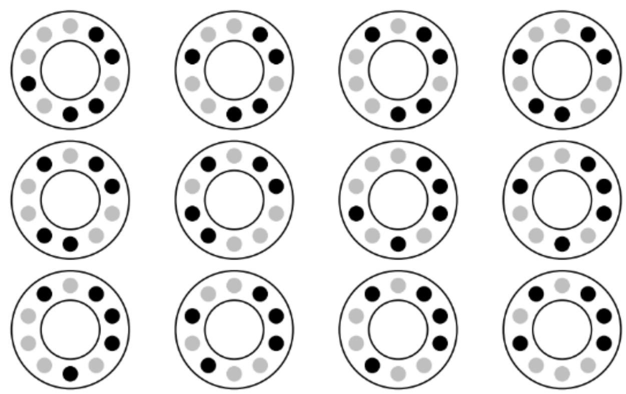
(1) If there is only one block, there is only one possible distribution. The five givers are contiguous, as are the five non-givers.



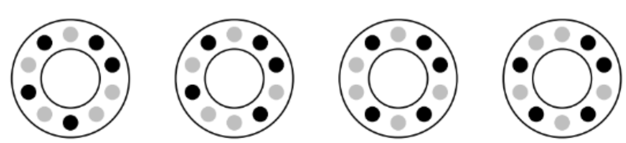
(2) If there are two blocks, there are two numerical partitions of 5 into 2 parts: $1+4$ and $2+3$. For each one one of those, we separate the two blocks with at least one of the 5 non-givers. In other words, we fill the blanks in B_1 __ B_2 __ with the 5 non-givers, with at least one in each blank. There are four ordered numerical partitions of 5 into two parts: $1+4, 2+3, 3+2$, and $4+1$, giving a total of $2 \cdot 4 = 8$ distributions with two blocks.



(3) With three blocks, there are two numerical partitions of 5 into three parts: $1+1+3$ and $1+2+2$. For each one, we fill the blanks in $B_1 \_\_\_ B_2 \_\_\_ B_3 \_\_\_$ with the 5 non-givers, with at least one in each blank. There are 6 ordered numerical partitions of 5 into three parts:$1+1+3, 1+3+1, 3+1+1, 1+2+2, 2+1+2$, and $2+2+1$, giving a total of $2 \cdot 6 = 12$ distributions with three blocks.



(4) With four blocks, there is only one numerical partition of 5 into four parts, namely $1+1+1+2$. We fill the blanks in $B_1 \_\_\_ B_2 \_\_\_ B_3 \_\_\_ B_4 \_\_\_$ with the 5 non-givers, with at least one in each blank. There are 4 ordered partitions of 5 into four parts: $1+1+1+2, 1+1+2+1, 1+2+1+1$, and $2+1+1+1$, giving a total of 4 distributions with four blocks.



(5) If there are five blocks, then there is only one distribution, where the givers strictly alternate with the non-givers.

Of these, only the case with 5 blocks has rotational symmetry, so we will consider that case separately. For the others, let B represent the number of blocks. From our discussion above, B counts the number of givers that are not receivers; thus $5 - B$ counts the number of digits that are both givers and receivers. Since there are 5 givers and 5 receivers, B also counts the number of receivers that are not givers, and $5 - B$ counts the number of digits that are neither givers nor receivers.

Because each giver is paired one non-giver, and exactly one element in each pair is a receiver, selecting the set of givers that are also receivers completely determines all four sets. For each possible value of B, the number of doubly great cycles with B blocks is given by

$$D(B)\binom{5}{5-B}B!(5-B)!B!(5-B)!,$$

where $D(B)$ is the number of distributions with B blocks, the binomial coefficient counts the number of ways of choosing digits that are both givers and receivers, and each of the factorials counts the number of ways to order each of the disjoint sets in the cycle.

For the remaining case where $B = 5$, the receivers must be $\{0, 1, 2, 3, 4\}$ and the cycle must strictly alternate between givers and receivers. There are 5!5! ways to order both sets, but each cycle is counted 5 times because of the rotational symmetry, so the number of doubly great cycles with 5 blocks is 5!4!. Thus, the total number os doubly great cycles is

$$1\binom{5}{4}(1!4!)^2 + 8\binom{5}{3}(2!3!)^2 + 12\binom{5}{2}(3!2!)^2 + 4\binom{5}{1}(4!1!)^2 + 5!4! = 46,080,$$

and the total number of doubly great numbers is $4 \cdot 46,080 = 184,320$.

We also provide the solution of the proposer. Define the "corresponding digit" of a digit a in a number N be the digit in N with the same place value as the digit a.

Define an *f-digit* as any digit, when multiplied by 2, will regroup. In other words, any digit from one to five. Then pairing f-digits to non f-digits based on the units digit of their double, the pairs would be 0-5, 1-6, 2-7, 3-8, and 4-9.

From here, each digit $d = 2q + r$ (where 2 is the divisor, q is the quotient, and r is the remainder) in $2G$ can be generated by letting the corresponding digit in G be either q or the f-digit q is paired with, $q + 5$, while r determines whether or not there is regrouping from the corresponding digit to the right of d (or another way to put it, whether or not this is a f-digit).

This means we could first place the f-digits in the last 9 places of G, then for any f-digit that isn't directly in front of another f-digit, its non f-digit pair would be placed directly in front of an f-digit. So to show that for any placement of the f-digits results in a valid placement, we just need to show the number of f-digits not in front of another f-digit is equal to the number of spaces directly in front of an f-digit after placing only f-digits.

Define a "clump" as any group of adjacent f-digits. Say there are c clumps in G. Then, for any clump, the rightmost f-digit in the clump is the only one not directly in front of another f-digit, and therefore there are exactly c such f-digits. The number of spaces directly in front of an f-digit is also c- the places in front of of clumps, as all the other spaces in front of an f-digit are already occupied by other f-digits. So now we have shown that the number of f-digits not in front of another f-digit is exactly equal to the number of empty spaces in front of an f-digit, and so for any placement of f-digits, we can place the non f-digits accordingly.

This means there obviously exists a solution (such as 1023456789), and multiple solutions.

We see we can split this into cases based on c, the number of clumps. We can first factor out the 5! for the order of the f-digits. Then, there will be $\binom{4}{c-1}$ possible ways to split the clumps into c clumps. Now, we can multiply by c! for the ways to permute the non f-digits directly to the left of clumps. Let such a digit be called *attached*.

Now, we will have c clumps with attached digits directly in front of them, along with $5-c$ non-attached non f-digits, so what is left to do is to arrange these 5 groups of digits. Because the groups of clumps already have an "order", we are just permuting the non attached digits, so we have ${}_5P_c$. After plugging in the values of c from one to five, we get $120+480+720+480+120=1920$, which after multiplying by $\frac{4}{5}$ to take away possibilities which start with zero, we get $\boxed{184320}$ as our final solution.

#1405: *Proposed by Steven J. Miller (Williams College), Rajaram Venkataramani and Anand Mohanram.* Let $p, p+2$ be odd twin primes at least 5; for example 5 and 7, 71 and 73, or 71,733,689 and 71,733,691. Multiply the two primes, and sum the digits. If the sum is not a one digit number, sum the digits again, and keep doing this until a one digit number arises. For example, for our three pairs we get 5 and 7 yields 35, so the digit sum is 8, while 71 and 73 has a product of 5,183 whose digit sum is 17 whose digit sum is 8, and the last pair's product is 5,145,722,281,016,099 is 62 which then gives a digit sum of 8. Is this a coincidence or will we always end with an 8?

Solution by Dylan Laramee and Daniel Podzunas, Western New England University. Also solved by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

We first show that if p, $p+2$ are odd twin primes at least 5, then their product x is congruent to 8 modulo 9. Notice that any integer must be expressed in the form of either $3n$, $3n+1$, or $3n+2$ for some $n \in \mathbb{N}$. Since $p = 3n \geq 5$ cannot be prime and $p = 3n+1$ implies $p+2 = 3n+3$ which cannot be prime, we know that p must be in the form of $3n+2$ for some $n \in \mathbb{N}$. Therefore,

$$x \;=\; p(p+2) \;=\; (3n+2)(3n+4) \;=\; 9n^2+18n+8 \equiv 8 \mod 9.$$

Secondly, we show that if $x \equiv 8 \mod 9$, then its repeated digital sum must be 8 as well. Let $x = \sum_{d=1}^{n} a_d \cdot 10^{d-1}$ with $a_1 \neq 0$ and $S(x) = \sum_{d=1}^{n} a_d$ be the digital sum of x. By observing

$$x \;=\; \sum_{d=1}^{n} a_d \cdot 10^{d-1} \;\geq\; \sum_{d=1}^{n} a_d \;=\; S(x),$$

we notice that $x > S(x) > (S \circ S)(x) > (S \circ S \circ S)(x) \ldots$ is strictly decreasing until the values drop to a single digit number. Let N be the first natural number such that $(\underbrace{S \circ \cdots \circ S}_{N\text{-times}})(x)$ is single-digit. Using the binomial theorem to expand 10^{d-1} as $(9+1)^{d-1}$ implies

$$x \;=\; \sum_{d=1}^{n} a_d \cdot (9+1)^{d-1} \;\equiv\; \sum_{d=1}^{n} a_d \cdot (1)^{d-1} \;=\; S(x) \mod 9$$

and therefore $x \equiv S(x) \equiv (S \circ S)(x) \equiv (S \circ S \circ S)(x) \equiv \ldots \equiv (\underbrace{S \circ \cdots \circ S}_{N\text{-times}})(x) \mod 9$.
Finally, we are able to conclude that $x \equiv 8 \mod 9$ implies

$$8 \equiv x \equiv (\underbrace{S \circ \cdots \circ S}_{N\text{-times}})(x) \mod 9$$

as needed.

GRE Practice #13:

GRE Practice #13: The following is inspired by a practice SAT problem, and the thought process of Cameron Miller in solving it. Let $f(x) = ax^4 + bx^2 + c$, with $f(0) = 1$, $f(1) = -3$ and $f(2) = 9$. What is $f(-1)$?
(a) -2 (b) -3 (c) -4 (d) -5 (e) -6.

The answer is (b) -3. We could find a, b and c as we have three equations and three unknowns; a little algebra (or linear algebra) yields $f(x) = 2x^4 - 6x^2 + 1$, and thus $f(-1) = 2-6+1 = -3$. A faster solution however is to note that $f(x) = f(-x)$ as we have f depends on only x^2 (i.e., it is an even function). Thus $f(-1) = f(1)$, and we are given $f(1) = -3$ and thus $f(-1) = -3$. The key point here, for many problems, focus on what you are asked to answer. Frequently there are many paths to a right answer. If we are going to evaluate f at many points, it is necessary to find the coefficients; however, if we just need it at one special point, while we can find the coefficients we may not need to. This is somewhat similar to some Lagrange multiplier problems, where often in solving $\nabla f(x_1, \ldots, x_n) = \lambda \nabla g(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n) = 0$ we do not need to find λ explicitly. It is needed so we have $n+1$ equations in $n+1$ unknowns, but often one takes ratios that eliminates the λ's as we progress to finding the candidates $(x_1, \ldots, x_n)$.

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