PI MU EPSILON: PROBLEMS AND SOLUTIONS: FALL 2024

STEVEN J. MILLER (EDITOR)

1. Problems: Fall 2024

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). The winning school from the Fall problem set is **Saint Bonaventure University**.

#1414: *Proposed by Kenneth Davenport.* Let γ be the Euler–Mascheroni constant, defined by

$$\gamma \ := \ \lim_{n\to\infty}\left(-\log n \ - \ \sum_{k=1}^{n}\frac{1}{k}\right).$$

Date: October 31, 2024.

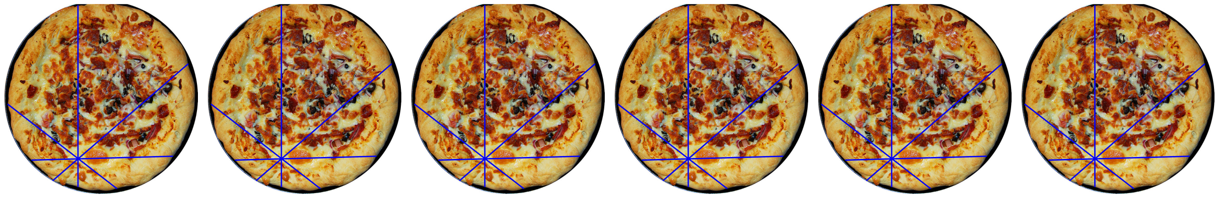

FIGURE 1. Pizza motivation; can you name the theorem that's represented here?

Prove that

$$\frac{1}{2^{n+1}} \;\le\; \gamma - \left[n\log 2 + \sum_{k=0}^{n-1} 2^{k-n+1} \sum_{j=0}^{2^k-1} \frac{1}{2j+1} \right] \;\le\; \frac{1}{2^n}.$$

In particular, we have a very accurate approximation for γ.

#1415: *Proposed by Kenneth Davenport.* Define the Pell numbers by $P_0 = 0, P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Prove or disprove: the sum of any 8 consecutive Pell numbers equals 24 times the fifth number in the sequence.

#1416: *Proposed by Serban Raianu, California State University, Dominguez Hills, and Joel Feldman, University of British Columbia. (Note: We are trying something new with this problem, namely having a long introduction to motivate* ***why*** *someone should be interested in this!)* This problem gives you an opportunity to win an integral solving competition against computer algebra systems.

Problem Q[25] of §3.3 in CLP-4 at https://personal.math.ubc.ca/*CLP/ asks for the evaluation of the surface integral

$$\iint_S xy^2 \, \mathrm{d}S,$$

where S is the portion of the sphere $x^2 + y^2 + z^2 = 2$ for which $x \ge \sqrt{y^2 + z^2}$. This can be easily integrated, e.g., by parametrizing S as the graph of a function $x = f(y, z)$, or by using the following parametrization in scrambled (y replacing x, z replacing y, and x replacing z) spherical coordinates:

$$\mathbf{r}(\phi,\theta) \;=\; \left\langle \sqrt{2}\cos(\phi)\,,\ \sqrt{2}\sin(\phi)\cos(\theta)\,,\ \sqrt{2}\sin(\phi)\sin(\theta) \right\rangle.$$

The solution to problem Q[25] of §3.3 in the book CLP-4, referenced above, warns that parametrizing S in standard spherical coordinates

$$\mathbf{r}(\phi,\theta) \;=\; \left\langle \sqrt{2}\sin(\phi)\cos(\theta)\,,\ \sqrt{2}\sin(\phi)\sin(\theta)\,,\ \sqrt{2}\cos(\phi) \right\rangle,$$

makes the evaluation of the integral very complicated. This happens because with the standard spherical coordinates parametrization the domain of the parameters is not rectangular, and we get one of the following intimidating double integrals:

$$
\begin{aligned}
I_1 &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{-\cos^{-1}\left(\frac{\csc(x)}{\sqrt{2}}\right)}^{\cos^{-1}\left(\frac{\csc(x)}{\sqrt{2}}\right)} \sin^4(x)\cos(y)\sin^2(y)\ \mathrm{d}y\ \mathrm{d}x, \\
I_2 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\sin^{-1}\left(\frac{\sec(y)}{\sqrt{2}}\right)}^{\pi-\sin^{-1}\left(\frac{\sec(y)}{\sqrt{2}}\right)} \sin^4(x)\cos(y)\sin^2(y)\ \mathrm{d}x\ \mathrm{d}y, \\
I_3 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\cos^{-1}\left(\frac{\sqrt{1-\tan^2(y)}}{\sqrt{2}}\right)}^{\pi-\cos^{-1}\left(\frac{\sqrt{1-\tan^2(y)}}{\sqrt{2}}\right)} \sin^4(x)\cos(y)\sin^2(y)\ \mathrm{d}x\ \mathrm{d}y.
\end{aligned}
$$

Computer algebra systems have trouble symbolically integrating these integrals, especially the last two, precisely because the integration domain is not rectangular. Can you evaluate I_1 and I_2 and I_3?

As a first hint, integration by parts can help in the evaluation of I_2 and I_3.

The last part of this problem can also be viewed as a second hint that might help you evaluate the three integrals above. Show that

$$
\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}\sin(x)(1-2\cos^2(x))^{\frac{3}{2}}\ \mathrm{d}x &= \int_0^{\frac{\pi}{4}} \frac{\tan^4(y)\sec^2(y)}{\sqrt{1-\tan^2(y)}}\ \mathrm{d}y \\
&= \int_0^{\frac{\pi}{2}} \sin^4(z)\ \mathrm{d}z, \qquad (1.1)
\end{aligned}
$$

by using just one substitution of the form

$$
\begin{aligned}
f(\textit{old variable}) &= g(\textit{new variable}) \\
f'(\textit{old variable})\ \mathrm{d}(\textit{old variable}) &= g'(\textit{new variable})\ \mathrm{d}(\textit{new variable})
\end{aligned} \qquad (1.2)
$$

where f and g are bijective functions, for each of the three pairs of integrals in (1.1). (The first integral appears in the computation of I_1, and the second integral appears in the computations of I_2 and I_3. This second part of this problem is not hard, the challenge here is to prove the three equalities of the pairs of integrals in (1.1) using just one substitution per equality.)

Here is some discussion about the substitution (1.2). Call the old variable x and the new variable u. Then the first equation of (1.2) implicitly defines the function $u(x)$ by requiring that $f(x) = g\big(u(x)\big)$ for all x, and the second equation of (1.2), $f'(x)\ \mathrm{d}x = g'(u)\ \mathrm{d}u$, is a memory aid which provides us with an easy way to remember that

$$
\int h\big(u(x)\big)\, f'(x)\ \mathrm{d}x = \int h(u)\, g'(u)\ \mathrm{d}u\Big|_{u=u(x)}
$$

This is much like, when, in the course of solving the separable differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\,g(y)$, we use the memory aid $\frac{\mathrm{d}y}{g(y)} = f(x)\ \mathrm{d}x$ as any easy way to remember that a function $y(x)$ which obeys

$$
\int \frac{\mathrm{d}y}{g(y)}\bigg|_{y=y(x)} = \int f(x)\ \mathrm{d}x
$$

also satisfies $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\,g(y)$.

#1417: *Proposed by Ivan Hadinata, Gadjah Mada University.* Consider any 2024 distinct positive integers $a_1 < a_2 < \cdots < a_{2024} \leq 293,335$. Define

$$A \ := \ \{(i,j) : i > j;\ i,j \in \{1,2,\cdots,2024\}\}$$

and

$$f(i,j) \ := \ |a_i - a_j| \text{ for all } (i,j) \in A.$$

Show that there are at least 8 distinct pairs $(i_1,j_1),\ (i_2,j_2),\ \cdots,\ (i_8,j_8) \in A$ such that

$$f(i_1,j_1) \ = \ f(i_2,j_2) \ = \ \cdots \ = \ f(i_8,j_8).$$

#1418: *Proposed by Steven J. Miller, Williams College.* Anyone who knows me well knows that I have a daily step challenge with a couple of my friends. My greatest month was averaging over 50,000 steps a day, which is the inspiration for this (and the next) problem. (a) Consider someone who averages exactly 50,000 steps in a 30 day month. Must there be a 20 day window (i.e., 20 consecutive days) where they walked at least 1 million steps? (b) What if we consider the days of the month to lie on a circle, so now day 30 is next to both days 29 and 1?

#1419: *Proposed by Steven J. Miller, Williams College.* Consider the framework of the previous problem, where someone walks on average 50,000 steps a day for a 30 day month. Is there a certain minimum steps per day, m, such that if they walk at least m steps a day then they must have walked at least one million steps during 20 consecutive days? If yes, what is the smallest m that works? Extra credit: if they walk on average A steps per day, with $A \geq 50,000$, what would m_A be to ensure they walk at least one million steps in a 20 day window?

GRE Practice #14: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy $f(f(x)) = 6x - f(x)$. Find $f(x)$. (This is a modification of Problem A5 from the 1988 Putnam Exan.)
(a) $6+x$ (b) $6x$ (c) $2x$ (d) $x-2$ (e) $3x$.

2. Solutions

#1407: *Proposed by Joe Santmyer, US Federal Government (retired).* Finding zeros of a function and their properties occupies a large literature in mathematics. Many solved and unsolved problems deal with zeros of a function. Notable statements, such as the Fundamental Theorem of Algebra and the Riemann Hypothesis, are center stage but many lesser known results are scattered in the literature.
The problem here was motivated by an exercise on page 155 in Stein and Shakarchi's *Princeton Lectures in Analysis II: Complex Analysis*, which is to prove that the entire function $f(z) = e^z - z$ has an infinite number of zeros. If an analytic function that is not identically zero has an infinite number of zeros then they are countable. Let $\{a_n\}$ be the sequence of zeros of f. What else can be said about the zeros? Prove one can at least say the following.

a. $\sum_{n=1}^{\infty} \frac{1}{a_n(1-a_n)} = 1$

b. $\sum_{n=1}^{\infty} \frac{1}{a_n(2\pi i m - a_n)} = 0$ where m is a nonzero integer

c. $\sum_{n=1}^{\infty} \frac{1}{a_n^2} = -1$

d. $\sum_{n=1}^{\infty} \frac{1}{a_n^3} = -\frac{1}{2}$.

Solution by Hongwei Chen, Christopher Newport University.

Since the growth order of $f(z)$ is 1 and $f(0) = 1 \neq 0$, by the Hadamard's factorization theorem (See Stein and Shakarchi's *Complex Analysis*, page 147), we have

$$e^z - z \;=\; e^{az+b} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n},$$

where a and b are constants.

Letting $z = 0$ gives $e^b = 1$ and thus we may take $b = 0$. Applying the logarithmic differentiation yields

$$\frac{e^z - 1}{e^z - z} \;=\; a + \sum_{n=1}^{\infty} \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right) \;=\; a + \sum_{n=1}^{\infty} \frac{z}{a_n(z - a_n)}. \tag{1}$$

Letting $z = 0$ in (1) gives $a = 0$. Moreover, letting $z = 1$ and $2\pi i m$ in (1), respectively yields (a) and (b) as claimed.

Next, differentiating (1) with respect to z gives

$$\frac{e^z(2 - z) - 1}{(e^z - z)^2} \;=\; -\sum_{n=1}^{\infty} \frac{1}{(z - a_n)^2}. \tag{2}$$

Now (c) follows by setting $z = 0$ in (2). Finally, differentiating (2) with respect to z gives

$$\frac{e^z(z^2 - 2z + 6) + e^{2z}(z - 3) - 2}{(e^z - z)^3} \;=\; \sum_{n=1}^{\infty} \frac{2}{(z - a_n)^3},$$

which implies (d) by letting $z = 0$.

#1409: *Proposed by Ron Evans and Steven J. Miller (Williams College).*
As mentioned in #1408, there is a long, rich history of trying to find which equations with integer coefficients have integer solutions (and if there are solutions, determining them). (a) Find all integer solutions to $3y^2 + 3y + 1 = x^3$. (b) Prove that $12x^3 - 3$ is never a square for $x > 1$.

Solution by Ivan Hadinata, Gadjah Mada University, Yogyakarta, Indonesia. Also solved by Kenny B. Davenport, and by Dylan Laramee of Western New England, and by Katherine Kuzniar of Saint Bonaventure University.
For the part (a), the only solutions are $(x, y) = (1, 0)$ and $(1, -1)$. It is trivial when $y = 0$ and $y = -1$, we get $(x, y) = (1, 0), (1, -1)$. Otherwise, observe that

$$x^3 = (y + 1)^3 - y^3. \tag{2.1}$$

The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is strictly increasing. By (2.1), it implies $x > 0$. If $y > 0$, by Fermat last theorem there will be no solution for (x, y). If $y < -1$, so let $y = -a$ where $a \in \mathbb{N}$ and $a \geq 2$, then (2.1) implies $x^3 + (a-1)^3 = a^3$. By Fermat last theorem, there is no satisfying solution (x, a). Thus, the only solutions are $(x, y) = (1, 0)$ and $(1, -1)$.

For part (b), suppose the contrary that there exists $b \in \mathbb{N}_0$ so that $12x^3 - 3 = b^2$ for some integer $x > 1$. Since $x > 1$, it implies $b > 3$. Since $12x^3 - 3$ is odd and divisible by 3, then b is in the form of $6k + 3$ for some $k \in \mathbb{N}$. Then

$$12x^3 - 3 = b^2 = 36k^2 + 36k + 9 \quad \Longrightarrow \quad x^3 = (k+1)^3 - k^3 \tag{2.2}$$

By Fermat last theorem, there is no solution $(x, k) \in \mathbb{N}^2$ satisfying (2.2). Thus $12x^3 - 3$ is not a square for all integers $x > 1$.

#1410: *Proposed by Kenny B. Davenport, St Petersburg, Florida.* The Pell numbers, defined by $P_0 = 0, P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$, are an interesting sequence of numbers with numerous properties; they are one of the simplest generalizations of the Fibonacci recurrence (same initial conditions but now $F_{n+1} = 1F_n + F_{n-1}$), and arise as the denominators in the sequence of the best rational approximations to $\sqrt{2}$. Not surprisingly, they satisfy a large number of interesting relations. Prove

$$2\sum_{k=1}^{n} kP_{k-1} \ = \ nP_{n+1} - (n+1)P_n.$$

Note: depending on the path you take to the proof, you may be able to generate many other additional identities, such as

$$2\sum_{k=1}^{n} k^2 P_{k-1} \ = \ (n^2+1)P_{n+1} - (n^2+2n)P_n - 1.$$

More generally, though you are only asked to prove the identity for the sum of k times P_{k-1}, can you conjecture what the shape of the answer should be for the sum of k^d times P_{k-1}?

Solution by G. C. Greubel, Newport News, VA. Also solved by Jackson Dry, St. Bonaventure University, NY, and by Dylan Laramee, Lauren Mauch, and Luke Stefaniak of Western New England.

First consider the series

$$\sum_{k=0}^{n} t^k \ = \ \frac{1 - t^{n+1}}{1-t}$$

and by differentiating with respect to t it is determined that

$$S_1(t) \ = \ \sum_{k=0}^{n} k\, t^{k-1} \ = \ \frac{1 - (n+1)\, t^n + n\, t^{n+1}}{(1-t)^2}$$

$$S_2(t) \ = \ \sum_{k=0}^{n} k^2\, t^{k-1} \ = \ \frac{1 + t - (n+1)^2\, t^n + (2n^2+2n-1)\, t^{n+1} - n^2\, t^{n+2}}{(1-t)^3}.$$

In the series $S_1(t)$ let $t = \{a^p, b^p\}$, where $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$, to obtain

$$\sum_{k=0}^{n} k\, a^{pn} = \frac{1 - (n+1)\, a^{pn} + n\, a^{pn+p}}{(1 - a^p)^2}$$
$$\sum_{k=0}^{n} k\, b^{pn} = \frac{1 - (n+1)\, b^{pn} + n\, b^{pn+p}}{(1 - b^p)^2}.$$

Using a general form $G_n = A\, a^n + B\, b^n$, with the property $G_{n+2} = 2\, G_{n+1} + G_n$, of the Pell and Pell-Lucas numbers the last set of series can be added in such a way that the result becomes

$$\sum_{k=0}^{n} k\, G_{pk+m} = \frac{1}{(1 + (-1)^p - Q_p)^2} \left[(G_{m+p} - 2\,(-1)^p\, G_m + G_{m-p}) \right.$$
$$-(n+1)\,(G_{p(n+1)+m} - 2\,(-1)^p\, G_{pn+m} + G_{p(n-1)+m})$$
$$\left. +n\,(G_{p(n+2)+m} - 2\,(-1)^p\, G_{p(n+1)+m} + G_{pn+m}) \right],$$

where Q_n is the n^{th} Pell-Lucas number. For the case of $p = 1$ the reduction yields

$$2 \sum_{k=0}^{n} k\, G_{k+m} = n\, G_{n+m+2} - (n+1)\, G_{n+m+1} + G_{m+1}.$$

Letting $G_n = \{P_n, Q_n\}$ gives the series:

$$2 \sum_{k=0}^{n} k\, P_{k+m} = n\, P_{n+m+2} - (n+1)\, P_{n+m+1} + P_{m+1}$$
$$2 \sum_{k=0}^{n} k\, Q_{k+m} = n\, Q_{n+m+2} - (n+1)\, Q_{n+m+1} + Q_{m+1}$$

and if $m = -1$ then

$$2 \sum_{k=0}^{n} k\, P_{k-1} = n\, P_{n+1} - (n+1)\, P_n$$
$$2 \sum_{k=0}^{n} k\, Q_{k-1} = n\, Q_{n+1} - (n+1)\, Q_n + 2.$$

Other series may be developed such as

$$2 \sum_{k=0}^{n} k\, G_{k+n} = (n-1)\, G_{2n+1} + n\, G_{2n} + G_{n+1}.$$

Now considering the same pattern for $S_2(t)$ the main result becomes

$$(1 + (-1)^p - Q_p)^3 \sum_{k=0}^{n} k^2\, G_{pk+m} = -n^2\, \phi_{n+3} + (2n^2 + 2n - 1)\, \phi_{n+2}$$
$$-\,(n+1)^2\, \phi_{n+1} + \phi_2 + \phi_1,$$

where

$$\phi_n = \sum_{j=0}^{3} \binom{3}{j} (-1)^{jp} G_{p(n-j)+m}.$$

When $p = 1$ the series reduces to

$$2 \sum_{k=0}^{n} k^2 G_{k+m} = (n^2+1) G_{n+m+2} - n(n+2) G_{n+m+1} - G_{m+2}.$$

Setting $G_n = \{P_n, Q_n\}$ and making choices for the value of m select series may be obtained such as:

$$\begin{aligned}
2 \sum_{k=0}^{n} k^2 G_{k+n} &= (n^2 - 2n + 1) G_{2n+1} + (n^2+1) G_{2n} - G_{n+2} \\
2 \sum_{k=0}^{n} k^2 P_{k-2} &= (n^2+1) P_n - n(n+2) P_{n-1} \\
2 \sum_{k=0}^{n} k^2 Q_{k-2} &= (n^2+1) Q_n - n(n+2) Q_{n-1} - 2 \\
2 \sum_{k=0}^{n} k^2 P_{k-1} &= (n^2+1) P_{n+1} - n(n+2) P_n - 1 \\
2 \sum_{k=0}^{n} k^2 Q_{k-1} &= (n^2+1) Q_{n+1} - n(n+2) Q_n - 2.
\end{aligned}$$

For the general case consider the series

$$S_{n,p}(t) = \sum_{k=0}^{n} k^p t^k$$

for which the following can be found

$$\begin{aligned}
S_{n,p}(t) &= S_{n+1,p}(t) - (n+1)^p t^{n+1} \\
&= \frac{t}{1-t} \left(-(n+1)^p t^{n+1} + \sum_{k=0}^{n} ((k+1)^p - k^p) t^k \right) \\
&= \frac{t}{1-t} \left(-(n+1)^p t^{n+1} + \sum_{j=0}^{p-1} \binom{p}{j} S_{n,j}(t) \right).
\end{aligned}$$

Letting $t = \{a^r, b^r\}$, multiplying by a^m, or b^m, respectively, and adding the results of the series it can be shown that

$$\begin{aligned}
(Q_r - 1 - (-1)^r) \sum_{k=0}^{n} k^p G_{rk+m} &= (n+1)^p (G_{r(n+1)+m} - (-1)^r G_{rn+m}) \\
&\quad - \sum_{j=0}^{p-1} \binom{p}{j} \sum_{k=0}^{n} k^j (G_{r(k+1)+m} - (-1)^r G_{rn+m}).
\end{aligned}$$

This is a recursive series which means that each series with the weight k^p needs to be calculated before k^{p+1} can be obtained. In the case $r = 1$ the series becomes

$$\begin{aligned} 2\sum_{k=0}^{n} k^p\, G_{rk+m} &= (n+1)^p\,(G_{n+m+1} + G_{n+m}) \\ &\quad - \sum_{j=0}^{p-1} \binom{p}{j} \sum_{k=0}^{n} k^j\,(G_{k+m+1} + G_{n+m}). \end{aligned}$$

When $r = p = 1$ then

$$\begin{aligned} 2\sum_{k=0}^{n} k\, G_{k+m} &= (n+1)\,(G_{n+m+2} - G_{n+m+1}) - \sum_{k=0}^{n}(G_{k+m+1} + G_{k+m}) \\ &= (n+1)\,(G_{n+m+2} - G_{n+m+1}) - G_{n+m+2} + G_{m+1} \\ &= n\,G_{n+m+2} - (n+1)\,G_{n+m+1} + G_{m+1}. \end{aligned}$$

When $r = 1$ and $p = 2$ then

$$\begin{aligned} 2\sum_{k=0}^{n} k^2\, G_{k+m} &= (n+1)^2\,(G_{n+m+2} - G_{n+m+1}) - \sum_{k=0}^{n}(G_{k+m+1} + G_{k+m}) \\ &\quad - 2\sum_{k=0}^{n} k\,(G_{k+m+1} + G_{k+m}) \\ &= (n^2+1)\,G_{n+m+2} - n\,(n+2)\,G_{n+m+1} - G_{m+2}. \end{aligned}$$

These series were presented earlier. The case of $r = 1$, and $p = 3$ gives

$$2\sum_{k=0}^{n} k^3\, G_{k+m} = (n^3+3n-3)\,G_{n+m+2} - (n^3+3n^2+1)\,G_{n+m+1} + G_{m+3} + G_{m+2}.$$

Further series can be developed in a similar pattern.

#1411: *Proposed by Joe Santmyer, US Federal Government (retired).*
With the aide of tables and technology (for example, *Mathematica*) show that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\frac{H_{j+1}}{j+1} = \frac{\pi^2}{12}.$$

Solution by Hongwei Chen, Christopher Newport University. Also solved by Kenny Davenport, and G. C. Greubel, Newport News, VA.

Let the proposed series be S. We first show that

$$\frac{H_{j+1}}{j+1} = -\int_0^1 x^j \ln(1-x)\,dx.$$

This follows from

$$\begin{aligned}\int_0^1 x^j \ln(1-x)\,dx &= \int_0^1 x^j \left(-\int_0^x \frac{dt}{1-t}\right)\,dx \\ &= -\int_0^1 \frac{1}{1-t}\left(\int_t^1 x^j\,dx\right)\,dt \;=\; -\frac{1}{j+1}\int_0^1 \frac{1-t^{j+1}}{1-t}\,dt \\ &= -\frac{1}{j+1}\int_0^1 (1+t+\cdots+t^j)\,dt \;=\; -\frac{H_{j+1}}{j+1}.\end{aligned}$$

Hence

$$\begin{aligned}S &= -\sum_{i=0}^{\infty}(-1)^i\left(\sum_{j=0}^{i}(-1)^j\binom{i}{j}\int_0^1 x^j\ln(1-x)\,dx\right) \\ &= -\sum_{i=0}^{\infty}(-1)^i\int_0^1\left(\sum_{j=0}^{i}(-1)^j\binom{i}{j}x^j\right)\ln(1-x)\,dx \\ &= -\sum_{i=0}^{\infty}(-1)^i\int_0^1 (1-x)^i\ln(1-x)\,dx \quad \text{(use the binomial theorem)} \\ &= -\sum_{i=0}^{\infty}(-1)^i\int_0^1 x^i\ln x\,dx \quad \left(\text{use } 1-x\to x \text{ and } \textstyle\int_0^1 x^n\ln x\,dx = -\frac{1}{(n+1)^2}\right) \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i}{(i+1)^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.\end{aligned}$$

#1412: *Proposed by Joe Santmyer, US Federal Government (retired)*
With the aide of tables and technology (for example, *Mathematica*) show that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\frac{h_{j+1}}{j+1} = [\sinh^{-1}(1)]^2.$$

Solution G. C. Greubel, Newport News, VA.

First note that this solution will require a generating function for the Harmonic numbers and the use of the dilogarithm function. The generating function for the Harmonic numbers, H_n, is given by, [1],

$$\sum_{n=1}^{\infty} H_n\,x^n = -\frac{\ln(1-x)}{1-x} \tag{2.3}$$

and the dilogarithm function, $\mathrm{Li}_2(x)$, is defined by, [2, 3],

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2} = -\int_0^x \frac{\ln(1-u)}{u}\,du = -\int_0^1 \frac{\ln(1-x\,t)}{t}\,dt. \tag{2.4}$$

Some particular evaluations and properties of the dilogarithm function are:

$$\begin{aligned}
\mathrm{Li}_2(0) &= 0 \\
\mathrm{Li}_2(1) &= \zeta(2) = \frac{\pi^2}{6} \\
\mathrm{Li}_2(-1) &= -\frac{\zeta(2)}{2} \\
\mathrm{Li}_2\left(\frac{1}{2}\right) &= \frac{\zeta(2)+\ln^2(2)}{2} \\
\mathrm{Li}_2 &= 2\left(\mathrm{Li}_2(x)+\mathrm{Li}_2(-x)\right).
\end{aligned} \tag{2.5}$$

A series that will be used is

$$\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}} \tag{2.6}$$

which can be derived by taking k derivatives, with respect to x, of both sides of the series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

A property of shifting of index of a series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n,k} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n+k,k}$$

will also be used.

Begin by considering the series

$$g(x) = \sum_{n=1}^{\infty} \frac{H_n}{n} x^n. \tag{2.7}$$

Differentiation of $g(x)$ leads to

$$\begin{aligned}
g'(x) &= \sum_{n=1}^{\infty} H_n x^{n-1} = -\frac{\ln(1-x)}{x(1-x)} \\
&= -\frac{\ln(1-x)}{x} - \frac{\ln(1-x)}{1-x},
\end{aligned}$$

where the generating function of the Harmonic numbers was used. Integrating both sides with respect to x, and making use of (2), leads to

$$\begin{aligned}
g(x) &= \int_0^x g'(u)\,du = -\int_0^x \frac{\ln(1-u)}{u}\,du - \int_0^x \frac{\ln(1-u)}{1-u}\,du \\
&= \left[\mathrm{Li}_2(u) + \frac{1}{2}\ln^2(1-u)\right]_0^x \\
&= \mathrm{Li}_2(x) + \frac{1}{2}\ln^2(1-x),
\end{aligned}$$

which can be stated as

$$g(x) \ = \ \sum_{n=1}^{\infty} \frac{H_n\, x^n}{n} \ = \ \mathrm{Li}_2(x) + \frac{1}{2}\, \ln^2(1-x). \tag{2.8}$$

Let $t \to -t$, in (6), to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, H_n\, t^n}{n} \ = \ \mathrm{Li}_2(-t) + \frac{1}{2}\, \ln^2(1+t). \tag{2.9}$$

Adding series (6) and (7) gives

$$\sum_{n=1}^{n} \frac{H_{2n}\, t^{2n}}{2\, n} \ = \ \mathrm{Li}_2(-t) + \mathrm{Li}_2(t) + \frac{\ln^2(1+t) + \ln^2(1-t)}{2}$$

or

$$\sum_{n=1}^{n} \frac{H_{2n}\, t^{2n}}{n} \ = \ \frac{1}{2}\, \left(\mathrm{Li}_2(t^2) + \ln^2(1-t^2) - 2\, \ln(1-t)\, \ln(1+t)\right). \tag{2.10}$$

Letting $t \to \sqrt{t}$, and making use of the property in (3), it is given that

$$\sum_{n=1}^{n} \frac{H_{2n}\, t^{n}}{n} \ = \ \frac{1}{2}\, \left(\mathrm{Li}_2(t) + \ln^2(1-t) - 2\, \ln(1-\sqrt{t})\, \ln(1+\sqrt{t})\right). \tag{2.11}$$

In this problem the series

$$h_n \ = \ \sum_{k=1}^{n} \frac{1}{2k-1} \ = \ H_{2n} - \frac{1}{2}\, H_n, \tag{2.12}$$

where H_n is the n^{th} harmonic number, is used. With this in mind, and with use of the harmonic number series (6) and (9), it is noted that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{h_n\, t^n}{n} \ &= \ \sum_{n=1}^{\infty} \left(H_{2n} - \frac{H_n}{2}\right)\, \frac{t^n}{n} \\ &= \ \frac{1}{4}\, \left(\ln^2(1-t) - 4\, \ln(1-\sqrt{t})\, \ln(1+\sqrt{t})\right). \end{aligned} \tag{2.13}$$

Now, consider the series

$$f(t) \ = \ \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \frac{h_{k+1}\, t^n}{k+1}. \tag{2.14}$$

in the following way:

$$\begin{aligned}
f(t) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\frac{h_{k+1}\,t^n}{k+1} \\
&= \sum_{n,k=0}^{\infty}\binom{n+k}{k}\frac{(-1)^n\,h_{k+1}\,t^{n+k}}{k+1} \quad \text{by use of index shifting} \\
&= \sum_{k=0}^{\infty}\frac{h_{k+1}\,t^k}{k+1}\cdot\sum_{n=0}^{\infty}\binom{n+k}{k}(-t)^n \\
&= \sum_{k=0}^{\infty}\frac{h_{k+1}\,t^k}{k+1}\cdot\frac{1}{(1+t)^{k+1}} \quad \text{by use of (4)} \\
&= \frac{1}{t}\sum_{k=0}^{\infty}\frac{h_{k+1}}{k+1}\left(\frac{t}{1+t}\right)^{k+1} \\
&= \frac{1}{t}\sum_{k=1}^{\infty}\frac{h_k}{k}\left(\frac{t}{1+t}\right)^{k}.
\end{aligned}$$

This may also be seen as, with the use of (11),

$$f(t) = \frac{1}{t}\sum_{k=1}^{\infty}\frac{h_k}{k}\left(\frac{t}{1+t}\right)^{k} = \frac{1}{t}\ln^2(\sqrt{1+t}+\sqrt{t})$$

and gives the expression

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\frac{h_{k+1}\,t^n}{k+1} = \frac{1}{t}\ln^2(\sqrt{1+t}+\sqrt{t}). \tag{2.15}$$

With this general form particular values of t may be considered. The specific value for the proposed problem is When $t=1$ for which

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\frac{h_{k+1}}{k+1} = \ln^2(1+\sqrt{2}) = \left(\sinh^{-1}(1)\right)^2.$$

Other series may also be obtained from the main result. For instance when $2\,t=-1$ it is determined that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{h_{k+1}}{2^n\,(k+1)} = -2\,\ln^2\left(\frac{1+i}{\sqrt{2}}\right) = -2\,\ln^2\left(e^{\pi i/4}\right) = \frac{3\,\zeta(2)}{4}$$

and when $4\,t=1$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\frac{h_{k+1}}{4^n\,(k+1)} = 4\,\ln^2(\alpha),$$

where $2\,\alpha = 1+\sqrt{5}$, α being the golden ratio. Another is the case for $t=-1$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{h_{k+1}}{k+1} = -\ln^2(i) = \frac{3\,\zeta(2)}{2}.$$

References

(1) E. D. Rainville, 'Infinite Series', Macmillan, 1967.
(2) L. Lewin, 'Dilogarithms and associated functions', London, Macdonald, 1958.
(3) L. Lewin, 'Polylogarithms and associated functions', New York, North-Holland, 1981.

GRE Practice #14: Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy $f(f(x)) = 6x - f(x)$. Find $f(x)$. (This is a modification of Problem A5 from the 1988 Putnam Exan.)
(a) $6 + x$ (b) $6x$ (c) $2x$ (d) $x - 2$ (e) $3x$.

The Putnam Problem asks for more – it asks to prove there is a unique function satisfying the conditions above. We could try to use advanced theory to find f, but there's no need. We have a very simple relation, there can only be one multiple choice answer, and all we have to do is check until we find one that works (or four that fail!). Thus, worse case, we have to try four of the options. If we try (a) we get

$$f(f(x)) \;=\; f(6+x) = 6 + (6+x) \;=\; 12 + x,$$

which is not $6x - (6+x) = 5x - 6$. Moving to (b) we find

$$f(f(x) \;=\; f(6x) \;=\; 6(6x) \;=\; 36x,$$

which is not $6x - 6x = 0$. Thus we continue to (c):

$$f(f(x)) \;=\; f(2x) \;=\; 2(2x) \;=\; 4x,$$

which is $6x - 2x = 4x$. Thus the answer is (c) and we only needed to check three items. Note it was very fast to check; take advantage of the answer being in front of you! For a short video on how to prove there is a unique function, see https://youtu.be/jZMoc3BrSZo.

Note we could check several possibilities at once by trying either ax (to handle (b), (c) and (e)) and $x + b$ (to handle (a) and (d)), or $ax + b$ to handle all three!

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