PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2025

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2025

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 20, while solutions for the Fall Issue must arrive by March 20 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). The winning school from the Fall problem set is **Western New England**.

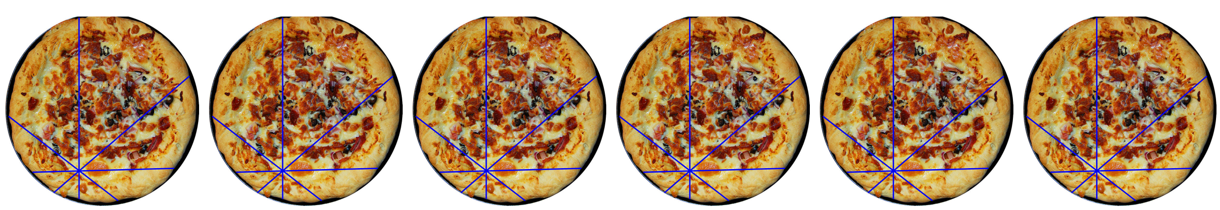


Figure 1. Pizza motivation; can you name the theorem that's represented here?

Date: April 1, 2025.

#1420: *Proposed by Etisha Sharma and Toyesh Prakash Sharma, Agra College.* Calculate

$$L \ = \ \lim_{n\to\infty} \sqrt[n]{\sin^2\left(\frac{\pi}{2n}\right)\sin^2\left(\frac{2\pi}{2n}\right)\cdots\sin^2\left(\frac{n\pi}{2n}\right)}.$$

As $\cos(x) = \sin(\pi/2 - x)$, we could replace the sines with cosines above.

#1421: *Proposed by Ivan Hadinata, Gadjah Mada University.* Find, with proof, all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which for all real x, y and z we have $f(x) + f(y + f(z)) = f(f(x) + z) + f(y)$.

#1422: *Proposed by Sourav Mandal (Ramakrishna Mission Vivekananda Educational and Research Institute) and Steven J. Miller (Williams College.)* Let σ_1 denote the sum of divisors function; thus as the divisors of 12 are 1, 2, 3, 4, 6 and 12 we have $\sigma_1(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Prove or disprove the following (note: the proposers have only proved the first, and leave the others as challenges to the readers; it is fine to just send partial solutions).

- There are infinitely many integers n such that $n = \sigma_1(a) + \sigma_2(b)$ for positive integers a and b.
- A positive percentage of all integers n can be written as $n = \sigma_1(a) + \sigma_2(b)$ for some positive integers a and b; in other words, there is some $p > 0$ that for all x sufficiently large, the number of $n \leq x$ that can be written as such a sum is at least px.
- Inspired by the Goldbach Problem, every sufficiently large integer n can be written as $n = \sigma_1(a) + \sigma_1(b) + \sigma_1(c)$ for positive integers a, b and c. (If the answer to this is yes, which numbers can be written as $\sigma_1(a) + \sigma_2(b)$?)

#1423: *Proposed by Steven Creech, Brown University* Given a positive integer $n > 1$, we have that n has a prime factorization given by

$$n \ = \ \prod_{i=1}^{r} p_i^{a_i}.$$

We recall that the function $\Omega(n) := \sum_{i=1}^{r} a_i$ counts the number of prime factors of n (with multiplicity). We have that a k-almost prime is a number n such that $\Omega(n) = k$. We note that a 1-almost prime is just a prime number while a 2-almost prime is known as a semi-prime. Now we wish to have a generalization of the prime counting function, $\pi(x) = \#\{n \leq x : n \text{ is prime}\}$ to the k-almost prime counting function

$$\pi_k(x) \ = \ \#\{n \leq x : \Omega(n) = k\}.$$

We will also define the square-free k-almost prime counting function as

$$\pi_k^*(x) \ = \ \#\{n \leq x : \Omega(n) = k, n \text{ is square-free}\}.$$

Now it was remarked in https://mathworld.wolfram.com/ that the following formula for $\pi_2(x)$ was discovered in 2005 by E. Noel and G. Panos, and then independently rediscovered by R.G. Wilson V. in 2006:

$$\pi_2(x) \;=\; \sum_{i=1}^{\pi(\sqrt{x})} \left(\pi\left(\frac{x}{p_i}\right) - i + 1\right). \tag{1.1}$$

where in the above formula we have that p_i denotes the i^{th} prime number.

However, the citation for both discoveries was given as personal communications and we could not find a formal proof of this formula in the literature. **Thus, prove** (1.1)**.**

More generally, one can also show

$$\pi_2^*(x) \;=\; \sum_{i=1}^{\pi(\sqrt{x})} \left(\pi\left(\frac{x}{p_i}\right) - i\right), \tag{1.2}$$

or even further one can prove the following.

Theorem 1.1. *For $k \geq 2$, we have the following formula for the number of k-almost primes less than x*

$$\pi_k(x) = \sum_{i_1=1}^{\pi(\sqrt[k]{x})} \sum_{i_2=i_1}^{\pi\left(\sqrt[k-1]{\frac{x}{p_{i_1}}}\right)} \sum_{i_3=i_2}^{\pi\left(\sqrt[k-2]{\frac{x}{p_{i_1}p_{i_2}}}\right)} \cdots \sum_{i_{k-1}=i_{k-2}}^{\pi\left(\sqrt{\frac{x}{p_{i_1}p_{i_2}\cdots p_{i_{k-2}}}}\right)} \left(\pi\left(\frac{x}{p_{i_1}p_{i_2}\ldots p_{i_{k-1}}}\right) - i_{k-1} + 1\right) \tag{1.3}$$

Similarly, we have the following formula for the number of square-free k-almost primes less than x

$$\pi_k^*(x) = \sum_{i_1=1}^{\pi(\sqrt[k]{x})} \sum_{i_2=i_1+1}^{\pi\left(\sqrt[k-1]{\frac{x}{p_{i_1}}}\right)} \sum_{i_3=i_2+1}^{\pi\left(\sqrt[k-2]{\frac{x}{p_{i_1}p_{i_2}}}\right)} \cdots \sum_{i_{k-1}=i_{k-2}+1}^{\pi\left(\sqrt{\frac{x}{p_{i_1}p_{i_2}\cdots p_{i_{k-2}}}}\right)} \left(\pi\left(\frac{x}{p_{i_1}p_{i_2}\ldots p_{i_{k-1}}}\right) - i_{k-1}\right) \tag{1.4}$$

#1425: *Proposed by Jacob Siehler, Gustavus Adolphus College* For which positive integers m can the set $\{0, 1, 2, \ldots, m-1\}$ be partitioned into triples $\{a, b, c\}$ with $a + b \equiv c \bmod m$?

#1424: *Proposed by Zhongxue Lü (Jiangsu Normal University) and Steven J. Miller (Williams College).* This problem is inspired from an observation in 2024, where the first proposer noticed that 2024 is formed by writing two multiples of four next to each other (in this case, we append to the right of $4 \cdot 5$ the product $4 \cdot 6$; as we had a backlog of problems we decided to wait till we hit the problem number ending in 24). More generally, let's consider all numbers of the form $4n(10^k + 1) + 4$ where k is the number of digits of $4(n + 1)$ and n is positive integer. We call such integers *appended 4's consecutive numbers.* How many such numbers are there less than 10^{100}?

GRE Practice #15:
Consider four independent, fair die, so each die lands on the numbers from 1 to 6 equally often (one-sixth of the time). What is the probability the sum of the four die is at most 5?
(a) 5/1296 (b) 1/81 (c) 2/81 (d) 1/27 (e) 1/9.

2. Solutions

Note: After the Fall 2022 issue went to press, we received some additional correct solutions and want to acknowledge the authors: Hyunbin Yoo from South Korea and Jeffrey Hemmelgarn at North Central College solved #1383 while his colleague Brennan Sweeney (also of North Central College) got Problem, #1387, while Sohom Dutta, DPS Ruby Park High School, Ian McDowell, University of South Carolina, Brian Bradie, Christopher Newport University, Hyunbin Yoo from South Korea and Rohan Dalal, The Episcopal Academy, the Eagle Problem Solvers of Georgia Southern University, the Cal Poly Pomona Problem Solving Group, and Thomas Reinke of Samford University all solved Problem #1388.

#1415: *Proposed by Kenneth Davenport.* Define the Pell numbers by $P_0 = 0, P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Prove or disprove: the sum of any 8 consecutive Pell numbers equals 24 times the fifth number in the sequence.

Solution by Henry Ricardo, Westchester Area Math Circle. Also solved by Panagiotis T. Krasopoulos, Kallithea, Athens, Greece, Emily Fountain and Dylan Laramee of Western New England (their solution is the one below), The Episcopal Academy Problem Solvers, the Skidmore College Problem Group and Jorge Montes and Sean Kanne (Cal Poly Pomona Problem Solving Group).

Consider the sum of any 8-consecutive Pell numbers from P_n to P_{n+7}. We first repeatedly apply the recursive definition to *break down* any term higher than P_{n+4} and *combine* any term lower than P_{n+4} as much as possible so that the expression is pushed toward the fifth term in the 8-consecutive sum.

$$\begin{aligned} & P_{n+7} + P_{n+6} + P_{n+5} + P_{n+4} \\ = & \; 3P_{n+6} + 2P_{n+5} + P_{n+4} \\ = & \; 8P_{n+5} + 4P_{n+4} \\ = & \; 20P_{n+4} + 8P_{n+3} \end{aligned} \qquad \begin{aligned} & P_{n+3} + P_{n+2} + P_{n+1} + P_n \\ = & \; 3P_{n+2} + 2P_{n+1} + P_n \\ = & \; 4P_{n+2} \end{aligned}$$

Using the two computations above in the 8-consecutive sum, we have

$$\begin{aligned} [P_{n+7} + P_{n+6} + P_{n+5} + P_{n+4}] + [P_{n+3} + P_{n+2} + P_{n+1} + P_n] &= 20P_{n+4} + 8P_{n+3} + 4P_{n+2} \\ &= 24P_{n+4} \end{aligned}$$

as claimed.

#1416: *Proposed by Serban Raianu, California State University, Dominguez Hills, and Joel Feldman, University of British Columbia. (Note: We are trying something new with this problem, namely having a long introduction to motivate* ***why*** *someone should be interested in this!)* This problem gives you an opportunity to win an integral solving competition against computer algebra systems.

Problem Q[25] of §3.3 in CLP-4 at https://personal.math.ubc.ca/*CLP/ asks for the evaluation of the surface integral

$$\iint_S xy^2 \, \mathrm{d}S,$$

where S is the portion of the sphere $x^2 + y^2 + z^2 = 2$ for which $x \geq \sqrt{y^2 + z^2}$. This can be easily integrated, e.g., by parametrizing S as the graph of a function $x = f(y, z)$, or by using the following parametrization in scrambled (y replacing x, z replacing y, and x replacing z) spherical coordinates:

$$\mathbf{r}(\phi,\theta) \;=\; \left\langle \sqrt{2}\cos(\phi)\,,\ \sqrt{2}\sin(\phi)\cos(\theta)\,,\ \sqrt{2}\sin(\phi)\sin(\theta) \right\rangle.$$

The solution to problem Q[25] of §3.3 in the book CLP-4, referenced above, warns that parametrizing S in standard spherical coordinates

$$\mathbf{r}(\phi,\theta) \;=\; \left\langle \sqrt{2}\sin(\phi)\cos(\theta)\,,\ \sqrt{2}\sin(\phi)\sin(\theta)\,,\ \sqrt{2}\cos(\phi) \right\rangle,$$

makes the evaluation of the integral very complicated. This happens because with the standard spherical coordinates parametrization the domain of the parameters is not rectangular, and we get one of the following intimidating double integrals:

$$\begin{aligned}
I_1 &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{-\cos^{-1}\left(\frac{\csc(x)}{\sqrt{2}}\right)}^{\cos^{-1}\left(\frac{\csc(x)}{\sqrt{2}}\right)} \sin^4(x)\cos(y)\sin^2(y)\,\mathrm{d}y\,\mathrm{d}x,\\
I_2 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\sin^{-1}\left(\frac{\sec(y)}{\sqrt{2}}\right)}^{\pi-\sin^{-1}\left(\frac{\sec(y)}{\sqrt{2}}\right)} \sin^4(x)\cos(y)\sin^2(y)\,\mathrm{d}x\,\mathrm{d}y,\\
I_3 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\cos^{-1}\left(\frac{\sqrt{1-\tan^2(y)}}{\sqrt{2}}\right)}^{\pi-\cos^{-1}\left(\frac{\sqrt{1-\tan^2(y)}}{\sqrt{2}}\right)} \sin^4(x)\cos(y)\sin^2(y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Computer algebra systems have trouble symbolically integrating these integrals, especially the last two, precisely because the integration domain is not rectangular. Can you evaluate I_1 and I_2 and I_3?

As a first hint, integration by parts can help in the evaluation of I_2 and I_3.

The last part of this problem can also be viewed as a second hint that might help you evaluate the three integrals above. Show that

$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}\sin(x)(1-2\cos^2(x))^{\frac{3}{2}}\,\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \frac{\tan^4(y)\sec^2(y)}{\sqrt{1-\tan^2(y)}}\,\mathrm{d}y\\
&= \int_0^{\frac{\pi}{2}} \sin^4(z)\,\mathrm{d}z, \qquad (2.1)
\end{aligned}$$

by using just one substitution of the form

$$\begin{aligned}
f(\textit{old variable}) &= g(\textit{new variable})\\
f'(\textit{old variable})\,\mathrm{d}(\textit{old variable}) &= g'(\textit{new variable})\,\mathrm{d}(\textit{new variable})
\end{aligned} \qquad (2.2)$$

where f and g are bijective functions, for each of the three pairs of integrals in (2.1). (The first integral appears in the computation of I_1, and the second integral appears in the computations of I_2 and I_3. This second part of this problem is not hard, the challenge here is to prove the three equalities of the pairs of integrals in (2.1) using just one substitution per equality.)

Here is some discussion about the substitution (2.2). Call the old variable x and the new variable u. Then the first equation of (2.2) implicitly defines the function $u(x)$ by requiring that $f(x) = g\big(u(x)\big)$ for all x, and the second equation of (2.2), $f'(x)\ \mathrm{d}x = g'(u)\ \mathrm{d}u$, is a memory aid which provides us with an easy way to remember that

$$\int h\big(u(x)\big)\ f'(x)\ \mathrm{d}x \ = \ \int h(u)\, g'(u)\ \mathrm{d}u\Big|_{u=u(x)}$$

This is much like, when, in the course of solving the separable differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\, g(y)$, we use the memory aid $\frac{\mathrm{d}y}{g(y)} = f(x)\ \mathrm{d}x$ as any easy way to remember that a function $y(x)$ which obeys

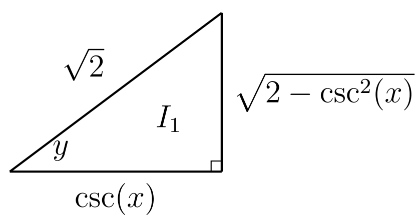
$$\int \frac{\mathrm{d}y}{g(y)}\Big|_{y=y(x)} \ = \ \int f(x)\ \mathrm{d}x$$

also satisfies $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\, g(y)$.

Solution by Luke Stefaniak, Western New England. Also solved by Charles Khu and Jaden Mueller (Cal Poly Pomona Problem Solving Group).

Our calculation shows $I_1 = I_2 = I_3 = \frac{\pi}{16}$.

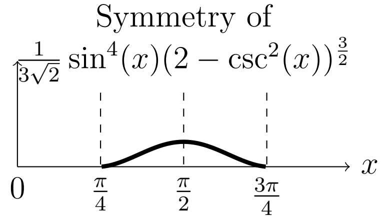
$$\begin{aligned}
I_1 &= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{-\arccos\left(\frac{\csc(x)}{\sqrt{2}}\right)}^{\arccos\left(\frac{\csc(x)}{\sqrt{2}}\right)} \sin^4(x)\cos(y)\sin^2(y)\, dy\, dx \\
&= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^4(x)\Big[\int_0^{\arccos\left(\frac{\csc(x)}{\sqrt{2}}\right)} 2\cos(y)\sin^2(y)\, dy\Big]\, dx \\
&= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^4(x)\Big[\frac{2}{3}\sin^3(y)\Big]_0^{\arccos\left(\frac{\csc(x)}{\sqrt{2}}\right)}\, dx \\
&= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{2}{3}\sin^4(x)\left(\frac{\sqrt{2-\csc^2(x)}}{\sqrt{2}}\right)^3 dx \\
&= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{3\sqrt{2}}\sin^4(x)(2-\csc^2(x))^{\frac{3}{2}}\, dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{3\sqrt{2}}\sin^4(x)(2-\csc^2(x))^{\frac{3}{2}}\, dx \\
&= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}\sin(x)(2\sin^2(x)-1)^{\frac{3}{2}}\, dx
\end{aligned}$$



$$y = \arccos\left(\frac{\csc(x)}{\sqrt{2}}\right)$$

$$\sin(y) = \frac{\sqrt{2-2\csc^2(x)}}{\sqrt{2}}$$

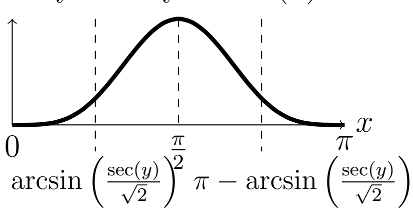
Symmetry of



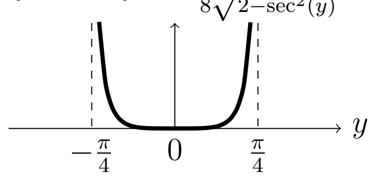
For I_2, we first establish the antiderivative of $\sin^4(x)$, symmetry of $\sin^4(x)$ and $\frac{4\tan^4(y)\sec^2(y)}{8\sqrt{2-\sec^2(y)}}$, and a triangle associating $x = \arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)$. Using $\int \sin^2(x)dx = -\frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}x + C$, integration by parts, and a standard algebra trick, we obtain an antiderivative of $\sin^4(x)$.

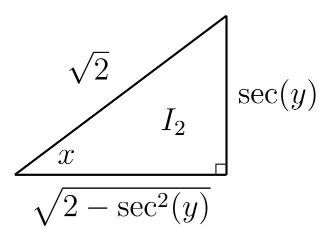
$$
\begin{aligned}
\int \sin^4(x)\,dx &= -\cos(x)\sin^3(x) + \int 3\cos^2(x)\sin^2(x)\,dx \\
&= -\cos(x)\sin^3(x) + 3\Big[\int \sin^2(x)\,dx\Big] - 3\int \sin^4(x)\,dx \\
&= -\frac{1}{4}\cos(x)\sin^3(x) - \frac{3}{8}\cos(x)\sin(x) + \frac{3}{8}x + C
\end{aligned}
$$

Symmetry of $\sin^4(x)$



Symmetry of $\frac{4\tan^4(y)\sec^2(y)}{8\sqrt{2-\sec^2(y)}}$





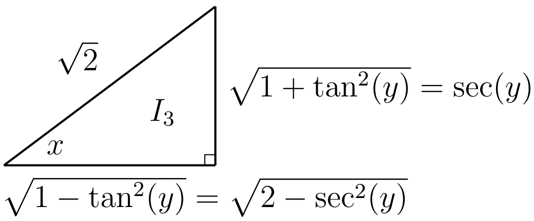
$$x = \arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)$$

$$\cos(x) = \frac{\sqrt{2-\sec^2(x)}}{\sqrt{2}}$$

$$\sin(x) = \frac{\sec(y)}{\sqrt{2}}$$

$$
\begin{aligned}
I_2 &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)}^{\pi-\arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)} \sin^4(x)\cos(y)\sin^2(y)\,dx\,dy \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(y)\sin^2(y)\Big[\int_{\arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)}^{\frac{\pi}{2}} 2\sin^4(x)\,dx\Big]\,dy \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(y)\sin^2(y)\Big[-\frac{1}{2}\cos(x)\sin^3(x) - \frac{3}{4}\cos(x)\sin(x) + \frac{3}{4}x\Big]_{\arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)}^{\frac{\pi}{2}}\,dy \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(y)\sin^2(y)\Big[\frac{3}{4}(\frac{\pi}{2} - \arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)) \\
&\qquad + \frac{1}{2}\frac{\sqrt{2-\sec^2(y)}}{\sqrt{2}}\left(\frac{\sec(y)}{\sqrt{2}}\right)^3 + \frac{3}{4}\frac{\sqrt{2-\sec^2(y)}}{\sqrt{2}}\frac{\sec(y)}{\sqrt{2}}\Big]\,dy \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos(y)\sin^2(y)\Big[\frac{3}{4}\arccos\left(\frac{\sec(y)}{\sqrt{2}}\right) + \frac{(\sec^3(y)+3\sec(y))\sqrt{2-\sec^2(y)}}{8}\Big]\,dy \\
&= \Big[\frac{1}{3}\sin^3(y)\left(\frac{3}{4}\arccos\left(\frac{\sec(y)}{\sqrt{2}}\right) + \frac{(\sec^3(y)+3\sec(y))\sqrt{2-\sec^2(y)}}{8}\right)\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \qquad (\textbf{Hint 1}) \\
&\quad - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{3}\sin^3(y)\left(\frac{3}{4}\frac{-\tan(y)\sec(y)}{\sqrt{2-\sec^2(y)}} + \frac{1}{8}\Big[(3\tan(y)\sec^3(y) + 3\tan(y)\sec(y))\sqrt{2-\sec^2(y)}\right. \\
&\qquad \left. + (\sec^3(y)+3\sec(y))\frac{-2\tan(y)\sec^2(y)}{2\sqrt{2-\sec^2(y)}}\Big]\right)\,dy \\
&= 0 - \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} -\frac{3\tan^2(y)\sin^2(y)}{4\sqrt{2-\sec^2(y)}} + \frac{1}{8}\Big[\frac{(3\tan^4(y)+3\tan^2(y)\sin^2(y))(2-\sec^2(y))}{\sqrt{2-\sec^2(y)}} \\
&\qquad - (\sec^3(y)+3\sec(y))\frac{\tan^3(y)\sin(y)}{\sqrt{2-\sec^2(y)}}\Big]\,dy \\
&= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{6\tan^2(y)\sin^2(y)}{8\sqrt{2-\sec^2(y)}} - \frac{6\tan^4(y)+6\tan^2(y)\sin^2(y) - 3\tan^4(y)\sec^2(y) - 3\tan^4(y)}{8\sqrt{2-\sec^2(y)}} \\
&\qquad + \frac{\tan^4(y)\sec^2(y) + 3\tan^4(y)}{8\sqrt{2-\sec^2(y)}}\,dy \\
&= \frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4\tan^4(y)\sec^2(y)}{8\sqrt{2-\sec^2(y)}}\,dy = \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\tan^4(y)\sec^2(y)}{\sqrt{2-\sec^2(y)}}\,dy
\end{aligned}
$$

For I_3, we establish a triangle associating $x = \arccos\left(\frac{\sqrt{1-\tan^2(y)}}{\sqrt{2}}\right)$. Using the identity $1+\tan^2(y) = \sec^2(y)$, we notice that the associated triangles of I_3 and I_2 are actually identical.



$$x = \arccos\left(\frac{\sqrt{1-\tan^2(y)}}{\sqrt{2}}\right) = \arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)$$

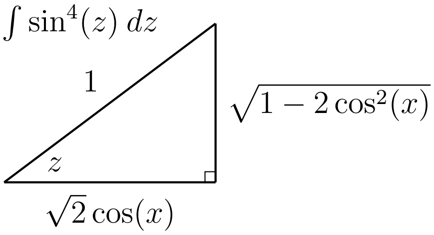
Since $\arccos\left(\frac{\sqrt{1-\tan^2(y)}}{\sqrt{2}}\right)$ and $\arcsin\left(\frac{\sec(y)}{\sqrt{2}}\right)$ are referring to the same angle, I_3 and I_2 share the same integrand and the same bounds, that is, $I_3 = I_2$.

Lastly, we follow (**Hint 2**) and show that I_1 and I_2 are both equivalent to

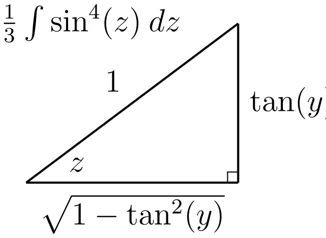
$$\frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^4(z)\,dz = \frac{1}{3}\Big[-\frac{1}{4}\cos(x)\sin^3(x) - \frac{3}{8}\cos(x)\sin(x) + \frac{3}{8}x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{16}$$

via one substitution for each. In particular, we use the following two associated triangles to demonstrate our trigonometric substitutions with trigonometric functions.

$I_1 \Rightarrow \frac{1}{3}\int \sin^4(z)\,dz$



$I_2 \Rightarrow \frac{1}{3}\int \sin^4(z)\,dz$



$$\sin(z) = \sqrt{1-2\cos^2(x)}$$
$$\cos(z) = \sqrt{2}\cos(x)$$
$$\sin(z)\,dz = \sqrt{2}\sin(x)\,dx$$

$$\cos(z) = \sqrt{1-\tan^2(y)}$$
$$\sin(z) = \tan(y)$$
$$\cos(z)\,dz = \sec^2(y)\,dy$$

$$\begin{aligned}
I_1 &= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}\sin(x)(1-2\cos^2(x))^{\frac{3}{2}}\,dx \\
&= \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \Big[(1-2\cos^2(x))^{\frac{3}{2}}\Big]\Big[\sqrt{2}\sin(x)\,dx\Big] \\
&= \frac{1}{3}\int_{\arccos(\sqrt{2}\cos(\frac{\pi}{4}))}^{\arccos(\sqrt{2}\cos(\frac{\pi}{2}))} \Big[\sin^3(z)\Big]\Big[\sin(z)\,dz\Big] \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^4(z)\,dz = \frac{\pi}{16}
\end{aligned}$$

$$\begin{aligned}
I_2 = I_3 &= \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\tan^4(y)\sec^2(y)}{\sqrt{1-\tan^2(y)}}\,dy \\
&= \frac{1}{3}\int_0^{\frac{\pi}{4}} \Big[\frac{\tan^4(y)}{\sqrt{1-\tan^2(y)}}\Big]\Big[\sec^2(y)\,dy\Big] \\
&= \frac{1}{3}\int_{\arcsin(\tan(0))}^{\arcsin(\tan(\frac{\pi}{4}))} \Big[\frac{\sin^4(z)}{\cos(z)}\Big]\Big[\cos(z)\,dz\Big] \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^4(z)\,dz = \frac{\pi}{16}
\end{aligned}$$

In addition, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}\sin(x)(1-2\cos^2(x))^{\frac{3}{2}}\,dx$ and $\int_0^{\frac{\pi}{4}} \frac{\tan^4(y)\sec^2(y)}{\sqrt{1-\tan^2(y)}}\,dy$ can be bridged via either $\sin(z)$ or $\cos(z)$, that is, $\tan(y) = \sqrt{1-2\cos^2(x)}$ or $\sqrt{2}\cos(x) = \sqrt{1-\tan^2(y)}$ respectively. We omit the routine calculation. □

#1418: *Proposed by Steven J. Miller, Williams College.* Anyone who knows me well knows that I have a daily step challenge with a couple of my friends. My greatest month was averaging over 50,000 steps a day, which is the inspiration for this (and the next) problem. (a) Consider someone who averages exactly 50,000 steps in a 30 day month. Must there be a 20 day window (i.e., 20 consecutive days) where they walked at least 1 million steps? (b) What if we consider the days of the month to lie on a circle, so now day 30 is next to both days 29 and 1?

Solution by Panagiotis T. Krasopoulos, Kallithea, Athens, Greece. Also solved by Rohan Dalal, The Episcopal Academy; by Karalyn Edwards and Dylan Laramee, Western New England; by Sean Kanne (Cal Poly Pomona Problem Solving Group); by Eagle Problem Solvers, Georgia Southern University,. The answer to (a) is no; we construct an example where every 20 day window is less than 1 million steps and still the average is exactly $50,000$ steps per day. Say that day 1 and day 30 have $190,000$ steps each and all the other days have $40,000$ steps. The total (30 days) sum of these steps equals

$$2 \times 190,000 + 28 \times 40,000 \;=\; 1,500,000.$$

The average is $1,500,000/30 = 50,000$, and observe that every 20 day window is less than 1 million. In fact the largest 20 day windows are the first (from day 1 to day 20) and the last (from day 11 to day 30), which both have $190,000 + 19 \times 40,000 = 950,000$ steps. All the other 20 day windows have $20 \times 40,000 = 800,000$ steps. Thus, we have an example where all the 20 day windows are less than 1 million steps and still the average is equal to $50,000$ steps per day.

For (b) we will see that it is impossible to have all the 20 day windows with less than 1 million steps and at the same time an average equal to $50,000$ steps per day; we must have a 20 day window with at least 1 million steps.

Let day i have x_i steps for $i = 1, \ldots, 30$ and assume that all the 20 day windows on the circle have less than 1 million steps; we will reach a contradiction. There are 30 such 20 day windows, which are

1 *to* 20,
2 *to* 21,
...
11 *to* 30,
12 *to* 31,
13 *to* 32,
...
30 *to* 49

(here the numbers greater than 30 are considered modulo 30).

We add the steps of all these 30 windows:

$$S \;:=\; \sum_{i=1}^{20} x_i + \sum_{i=2}^{21} x_i + \cdots \sum_{i=11}^{30} x_i + \sum_{i=12}^{31} x_i + \sum_{i=13}^{32} x_i + \cdots \sum_{i=30}^{49} x_i,$$

where again numbers greater than 30 are considered modulo 30. We are assuming that these 30 windows have less that 1 million steps each, and also we observe that in S every x_k appears exactly 20 times. Thus

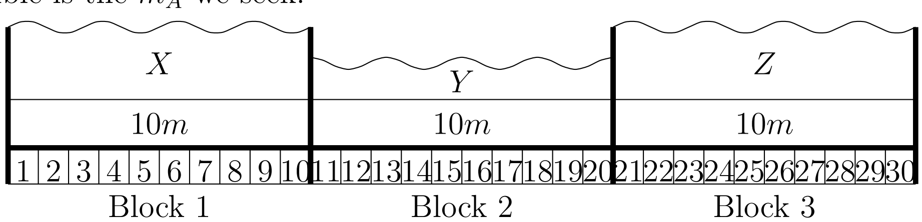
$$S \;=\; 20\sum_{i=1}^{30} x_i < 30 \times 1,000,000 \;\Leftrightarrow\; \sum_{i=1}^{30} x_i < 1,500,000.$$

The average is $\sum_{i=1}^{30} x_i/30 < 50,000$, which means that if we want an average of $50,000$ steps or more per day, there is at least one 20 day window with at least 1 million steps. □

#1419: *Proposed by Steven J. Miller, Williams College.* Consider the framework of the previous problem, where someone walks on average 50,000 steps a day for a 30 day month. Is there a certain minimum steps per day, m, such that if they walk at least m steps a day then they must have walked at least one million steps during 20 consecutive days? If yes, what is the smallest m that works? Extra credit: if they walk on average A steps per day, with $A \geq 50,000$, what would m_A be to ensure they walk at least one million steps in a 20 day window?

Solution by Tiba Draia and Dylan Laramee, Western New England. Also solved by Sean Kanne (Cal Poly Pomona Problem Solving Group), and by the Eagle Problem Solvers, Georgia Southern University,. We argue that $m = 50,000$ is necessary by ruling out its predecessor. Suppose $m = 49,999$. We have the freedom to distribute the remaining 30 steps to reach a total of 1.5 million steps. If days 1 and 30 each receive a half of the 30 steps, resulting in $50,014$ steps on both days, it is easy to see that there is no 20-day window in which you would walk more than 1 million steps.

Fix a total of $30A$ steps over a month. After allocating $m \leq A$ steps into each day, our goal is to distribute the remaining $30(A-m) = X + Y + Z$ steps while keeping the *highest step count* across all 20-day windows under a million. The smallest m for which this is no longer possible is the m_A we seek.



We base our 3-Block strategy on the following analysis.

- Distributing Y steps anywhere within Block 2 must result in raising the step count in *all* the 20-day windows by Y.
- Distributing X steps anywhere within Block 1 must result in raising the step count in the *first* 20-day window by X but leaves the *last* 20-day window unaffected. Symmetrically, the same applies to Block 3.
- Without loss of generality, we place Z on the last day of the month to prevent overlap with the 20-day windows that already include values from X.

Now, the *highest step count* across all 20-day windows can be expressed as

$$\max\{X + Y + 20m, Y + Z + 20m\}.$$

In order to minimize it, Y obviously should be zero since the steps in Block 2 are counted twice. Moreover, X and Z should evenly split $30(A - m_A)$. In particular,

$$\min_{X+Y+Z=30(A-m)} \max\{X + Y + 20m, Y + Z + 20m\} = \frac{30(A - m)}{2} + 20m = 15A + 5m$$

As we require $15A + 5m < 1,000,000$ and $A \geq 50,000$, the smallest m that violates the inequality is $200,000 - 3A$. Therefore, $m_A = 200,000 - 3A$.

In addition, $m_{50,000} = 50,000$ confirms our solution to the original problem.

GRE Practice #15:
Consider four independent, fair die, so each die lands on the numbers from 1 to 6 equally often (one-sixth of the time). What is the probability the sum of the four die is at most 5? (a) 5/1296 (b) 1/81 (c) 2/81 (d) 1/27 (e) 1/9.

Solution: We can compute the answer without too much trouble: it is $\binom{4}{4}(1/6)^5 + \binom{4}{1}\binom{3}{3}(1/6)^5$ (the first term is the probability we have all 1's, the second term the probability we have three 1's and one 2; these are the only options). Note however that the probability we roll a 1 or a 2 on any die is $2/6 = 1/3$, thus the probability that all the die are either a 1 or a 2 is $(1/3)^4 = 1/81$. This is clearly a strict upper bound for the probability the sum is at most 5 (as we cannot have four 2's for example); thus the answer is (a).

Email address: sjm1@williams.edu

Professor of Mathematics, Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267