PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2026

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2026

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to \$100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 20, while solutions for the Fall Issue must arrive by March 20 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). Unfortunately this cycle no school had at least two correct solutions.

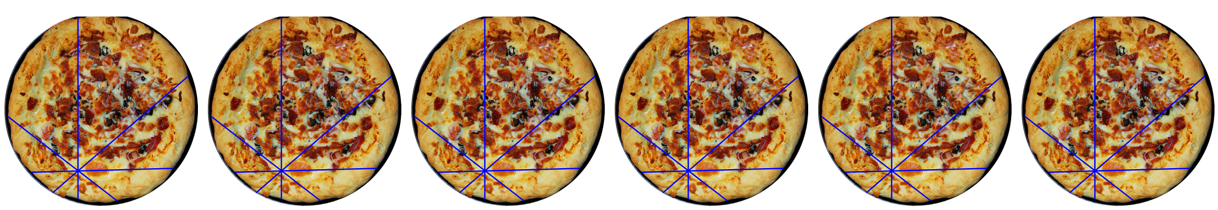

Figure 1. Pizza motivation; can you name the theorem that's represented here?

Date: April 14, 2026.

Each year a distinguished mathematician gives the PME J. Sutherland Frame Lecture at the Joint Mathematics Meetings. For 2026 the speaker was Francis Su from Harvey Mudd College. We are delighted to start the problem section off with a contribution from him based on his talk.

#1431: *Proposed by Francis Su (Harvey Mudd College).* This problem is an outgrowth of his talk *100 Years of Sperner's Lemma: Proofs, Generalizations, and Applications*, which was the 2026 PME J. Sutherland Frame Lecture delivered at the Joint Mathematics Meetings on January 4th, 2026 in Washington, D.C.; it is available online at `https://youtu.be/dPXZZJyWw2I`.

Prove a version of Sperner's lemma for squares, which goes as follows. Suppose you have a square that is triangulated (which means subdivided into triangles so that pairs of triangles intersect each other only at a vertex of both, or on an edge of both, or not at all). Vertices in this triangulation are labelled using the following rule: the corner vertices of the original square are labeled 1, 2, 3, and 4. Each vertex on the edge between the corner vertices labeled i and j must be labelled either i or j. All other vertices (on the inside of the square) are labeled by any of the labels 1, 2, 3, 4. Show that there must be at least 2 triangles that each have 3 different labels. Note if you divide the square into squares, however, this need not be true.

#1432: *Proposed by El Houcein El Abdalaoui, Université de Rouen Normandie and Steven J. Miller, Williams College.* Find all positive integers $B \geq 2$ such that there are infinitely many even numbers x such that the sum of the digits of x's base B expansion is an odd prime? For which B can you do it where none of each x's digits are zero?

#1433: *Proposed by El Houcein El Abdalaoui, Université de Rouen Normandie and Steven J. Miller, Williams College.* We say n is *four-reciprocal-triple* good if there are positive integers x, y and z such that

$$\frac{4}{n} \;=\; \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Prove that as $N \to \infty$ the percentage of positive integers $n \leq N$ that are four-reciprocal triple good is at least 50%. (For additional fun: Write some code and investigate; what do you think the limit is as $N \to \infty$? This is a special case of a famous conjecture.)

#1434: *Proposed by El Houcein El Abdalaoui, Université de Rouen Normandie and Steven J. Miller, Williams College.* Frequently in number theory a problem is significantly easier if we allow signs; thus instead of consider the problem as stated in #1434 ($4/n = 1/x+1/y+1/z$ with x, y, z positive integers) we instead investigated the signed variant $4/n = 1/x+1/y+1/z$ with $xyz \neq 0$. Prove there is always a solution for any integer $n > 1$.

#1435: *Proposed by Charles Callender, Johns Hopkins University, and Steven J. Miller, Williams College.*

This problem is adapted from *Valuing Pharmaceutical Assets: When to Use NPV vs rNPV*, available at https://www.alacrita.com/whitepapers/valuing-pharmaceutical-assets-when-to-use-npv-vs-rnpv.

Our goal is to highlight how math you've seen for years has enormous applications in the real world. Specifically, how do major pharmaceutical companies determine which drugs are worth investment? This is a tremendously challenging problem, as they need to know what the revenue will be if successful (how much and for how many years), as well as know the likelihood for success! If they can accurately estimate all these numbers, by investing in a large number of drugs the central limit theorem kicks in and they can be reasonably confident about their revenues.

Net Present Value (NPV) defines how much an investment is worth over its lifetime, discounted to today's value. NPV can be calculated as follows:

$$\text{NPV} = \sum_{t=0}^{T} \frac{R_t}{(1+i)^t}$$

where

- R_t = net cash flow at time period t,
- i = discount rate,
- t = time period, with $t = 0$ representing today,
- T = total number of time periods.

Consider the following problem. Vaccines Inc. is considering investing in a research and development (R&D) project for a new drug that involves multiple stages. The project details are as follows.

- **Stage 1 (Drug Development):**
  - **Duration:** 1 year (Year 1).
  - **Investment:** \$5 million at the beginning of Year 1 ($t = 0$).
  - **Probability of Success:** p_1 (where $0 < p_1 < 1$).
  - **Cash Flows:** No cash inflows during this stage.
- **Stage 2 (Clinical Trials):**
  - **Duration:** 1 year (Year 2).
  - **Investment:** \$10 million at the beginning of Year 2 ($t = 1$), but only if Stage 1 is successful.
  - **Probability of Success:** p_2 (where $0 < p_2 < 1$).
  - **Cash Flows:** No cash inflows during this stage.
- **Stage 3 (Commercialization and Patent Period):**
  - **Duration:** 1 year (Year 3).
  - **Investment:** No additional investment required.
  - **Annual Revenue:** \$X million received at the end of Year 3 ($t = 3$).
  - **Probability of Reaching Stage 3:** $p_1 \times p_2$.
- **Stage 4 (Post-Patent Period):**
  - **Duration:** 1 year (Year 4).
  - **Annual Revenue:** $\frac{X}{2}$ million received at the end of Year 4 ($t = 4$).
  - **Investment:** No additional investment required.
- **Discount Rate:** 9% per year.

Additional Assumptions:

- Investments are made at the *beginning* of each respective stage.
- Revenues are received at the *end* of each year during Stages 3 and 4.
- All cash flows should be discounted back to $t = 0$ using the discount rate of 9%.
- If Stage 1 or Stage 2 is not successful, the company immediately discontinues the project and no further investments are made.
- The probabilities p_1 and p_2 reflect the company's best estimate of success at each stage, based on historical data, clinical evidence, and expert judgment.

Questions:

(a) Calculate the NPV of the entire project as a function of X, p_1, and p_2, taking into account the probabilities of success at each stage.

(b) Determine the value of X (for a given p_1, p_2) that makes the company indifferent to pursuing the drug's development (i.e., find X such that the expected NPV of the project is zero).

Important Real-World Considerations: As remarked, these calculations depend heavily on accurate estimation of various inputs.

- **Success Probabilities:** Estimating the probability of success at each stage (p_1, p_2, etc.) is challenging. Companies rely on historical data, clinical evidence, and expert judgment, but these estimates are subject to considerable uncertainty.
- **Revenue Projections:** Forecasting future revenues (X) involves uncertainty about market size, competition, pricing, and future regulations. Even with current estimates, unforeseen events, like new market entrants and policy shifts, can drastically change financial outcomes.
- **Data Availability and Quality:** In reality, companies may have incomplete or unreliable information. They must invest in research, consultations, and market analyses to refine these estimates. The better the information, the more reliable the valuation.

Comment: Although this exercise shows how to apply NPV under given assumptions, real-world valuations are significantly more complex. Accurately estimating probabilities of success at each development stage, projecting market sizes and future revenues, and choosing appropriate discount rates typically involves substantial research, expert input, careful statistical analysis, and continuous refinement. What if we do not have the correct information? Another company could bring a competing product to market earlier than expected, reducing potential profits; development timelines might stretch longer than planned; or probabilities of success might be overly optimistic. Such uncertainties can force a reassessment of a project's value and underscore the importance of building flexible models that can be revised as more information becomes available. One has to explore methods for estimating these success probabilities and revenue projections, comparing different market assumptions, and performing sensitivity analyses to understand how uncertainties in these parameters influence the decision to move forward.

GRE Practice #17: Let G be a group with at most 30 elements and consider its action on itself by conjugation: given $x \in G$ the action of $g \in G$ on x is defined by $g^{-1}xg$. Assume for each x the stabilizer is just $\{e\}$, the identity element. How many such groups are there modulo 5?.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 5.

2. Solutions

#1426: *Proposed by Robert Laniewski, Cape Town, South Africa.* Fermat's Last Theorem states that there are no integer solutions $a^n + b^n = c^n$ where a, b, and c are positive integers, and n is any integer greater than 2. In 2015, and then revised in 2020, a paper posted on arXiv[1] observed that there are no solutions to the Fermat equation where some of the components are prime. Specifically, there are no integer solutions where $a + b$, b, or c are prime, assuming that $c > b > a$, and $n > 2$ is any odd integer. Independently, we have found a one-page proof demonstrating that there are no integer solutions to the Fermat equation where a and b are both prime, and $n > 2$ is any odd integer (with no specific stipulation that $b > a$). Thus, prove elementarily that there are no solutions with a and b both non-zero positive prime numbers.

Solution by Panagiotis T. Krasopoulos Kallithea, Athens, Greece. Also solved by Sixu Liu and Yuntong Xiao at Mount Holyoke College. We assume that there is a solution for primes $a, b > 1$ and an integer $n > 1$, and so we have

$$a^n \;=\; c^n - b^n \;=\; (c-b)S_n,$$

where

$$S_n \;=\; \sum_{k=0}^{n-1} c^{n-1-k}b^k.$$

Now since a is prime it must be $c - b = a^m$ and $S_n = a^{n-m}$ for an integer $m \geq 0$. But if $m > 0$ then we have $c = b + a^m$ and so

$$a^n + b^n \;=\; (b + a^m)^n.$$

Also from the binomial theorem we know that

$$(b + a^m)^n \;=\; b^n + a^{mn} + \sum_{k=1}^{n-1} \binom{n}{k} b^{n-k}a^{mk} > b^n + a^n,$$

and we have a contradiction as $(b + a^m)^n = c^n$ equals $a^n + b^n$. Thus, $m = 0$ and so $c = b + 1$. By doing the same analysis starting from $b^n = c^n - a^n = (c-a)S_n^{'}$, we similarly conclude that $c = a + 1$.

Since we have found that $c = b + 1$ and $c = a + 1$, the only possible case is when $a = b$. For this case we get

$$a^n + a^n \;=\; c^n,$$

[1] Yu-Lin Chou, *Fermat's Equation Has No Solution with Some Prime Components*, https://arxiv.org/abs/1505.02457

or

$$\sqrt[n]{2} = \frac{c}{a}.$$

But this is impossible since $\sqrt[n]{2}$ is an irrational number for $n \geq 2$ and c/a is a rational number. Thus, we have a contradiction and such a solution does not exist.

A final remark is that in our proof we did not assume that $n > 1$ is an odd number. This fact tell us also that there are no Pythagorean triples (a, b, c) with a, b primes. On the other hand we know from e.g. the Pythagorean triples $(3, 4, 5)$, $(5, 12, 13)$ that a and c (or symmetrically b and c) can indeed be primes.

#1427: *Proposed Cameron, Elizabeth, Kayla and Steven Miller.* While admiring the numerous bridges in Venice on a water taxi, the four of us passed the time discussing another type of bridge, the card game, which led to the following interesting problem. *What is the fewest number of points you and your partner can have to make 7 spades under the following conditions: (i) you may distribute the cards however you want, but (ii) you must assume intelligent play by the opponents.*

Solution by Jack Boge, Skidmore College. Also solved by Kenny B. Davenport and John Gergel. See Figure 2 for a deal that works.

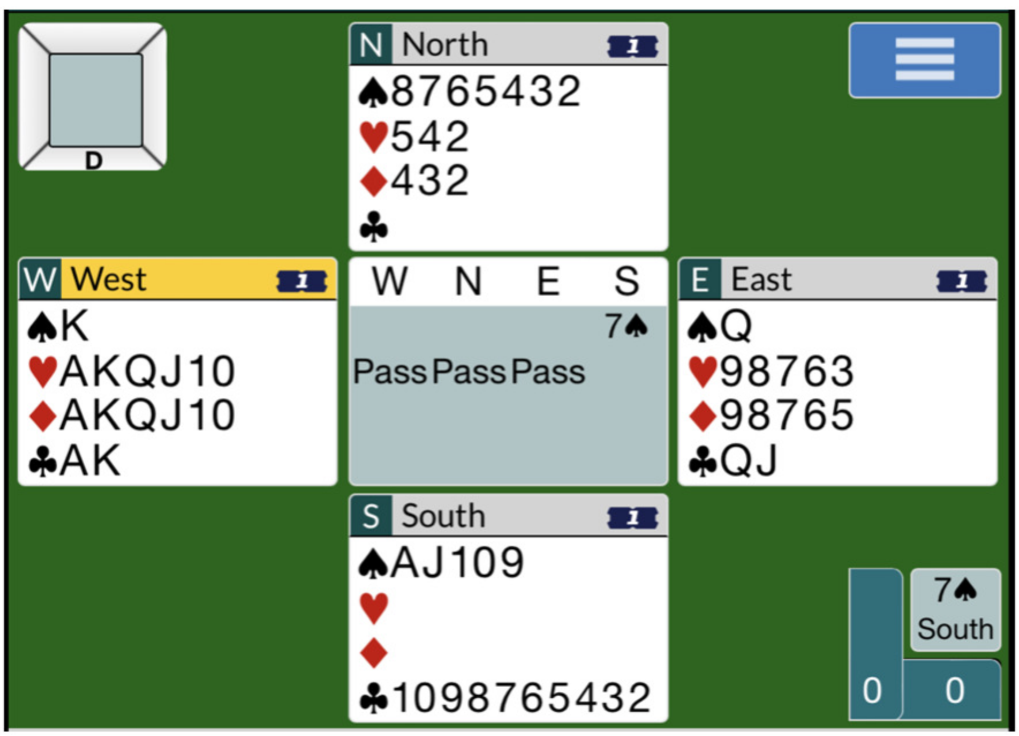

Figure 2. A solution with just five points in North - South.

On a red suit lead, South trumps, cashes the ace of spades, trumps a club, plays a spade back to hand, trumps a club, and now crosses back to hand and is making. On a club lead, South performs the same process except one less time. On a spade lead, the play is trivially similar.

It is not possible to do with four or fewer points. Clearly North-South must have the Ace of spades, else they have a loser. Is it possible to only have the Ace? If they only have the Ace then they are missing the King, Queen and Jack, and clearly either East or West has at least two of the three. East-West must make one of them (as the Ace can only beat one of them, and thus either the other is a winner, or falls potentially on an honor of its partner if the partner has at least two trump).

#1428: *Proposed by Hongwei Chen, Christopher Newport University.* In the latest edition of Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* (8th ed., 2007), Entry 3.688.14 states that

$$\int_0^{\pi/2} \frac{1}{(\tan^\mu x + \cot^\mu x)^\nu} \frac{dx}{\tan x} \;=\; \frac{\sqrt{\pi}}{2^{2\nu+1}\mu} \frac{\Gamma(\nu)}{\Gamma(\nu+1/2)} \quad (\nu > 0). \tag{1.}$$

A few tests for parameters μ and ν by Mathematica indicate (1) is incorrect. Can you offer a correction to (1)?

Motivation. Formula (1) caught my attention when I worked on Monthly Problem 12407 and found that, for $r > 0$,

$$\int_0^{\infty} \frac{x^{r-1}}{(1+x^2)(1+x^{2r})}\, dx \;=\; \frac{\pi}{4r}. \tag{2}$$

The change of variable $x \to \tan x$ converts (2) into

$$\int_0^{\pi/2} \frac{1}{(\tan^r x + \cot^r x)} \frac{dx}{\tan x} \;=\; \frac{\pi}{4r},$$

which is different from the result obtained from (1). A few tests for other parameters μ and ν by Mathematica also indicate (1) is incorrect.

Solution by Seán M. Stewart, King Abdullah University of Science and Technology. Let

$$I(\mu,\nu) \;=\; \int_0^{\pi/2} \frac{1}{(\tan^\mu x + \cot^\mu x)^\nu} \frac{dx}{\tan x}.$$

If $\mu = 0$, then

$$\begin{aligned} I(0,\nu) &= \frac{1}{2^\nu}\int_0^{\pi/2} \cot(x)\, dx \;=\; \frac{1}{2^\nu} \lim_{a\to 0^+} \int_a^{\pi/2} \cot(x)\, dx \\ &= \frac{1}{2^\nu} \lim_{a\to 0^+} \left[\log(\sin x)\right]_a^{\pi/2} \;=\; -\frac{1}{2^\nu} \lim_{a\to 0^+} \log(\sin a) \;=\; \infty. \end{aligned}$$

Since this limit does not exist, $\mu \neq 0$.

Consider $\mu > 0$. Enforcing a substitution of $u = \tan x$, $dx = \frac{du}{1+u^2}$, while the limits of integration are mapped from $(0, \frac{\pi}{2}) \mapsto (0, \infty)$. So for the integral we have

$$I(\mu,\nu) \;=\; \int_0^{\infty} \frac{1}{\left(u^\mu + \frac{1}{u^\mu}\right)^\nu} \frac{du}{u(1+u^2)} \;=\; \int_0^{\infty} \frac{u^{\mu\nu-1}}{(u^{2\mu}+1)^\nu} \frac{du}{1+u^2}. \tag{2.1}$$

Next, enforcing a substitution of $u \mapsto \frac{1}{u}$, $du \mapsto -\frac{du}{u^2}$, while the limits of integration are mapped from $(0,\infty) \mapsto (\infty, 0)$. Thus

$$\begin{aligned} I(\mu,\nu) &= \int_0^\infty \frac{1}{u^{\mu\nu-1}\left(\frac{1}{u^{2\mu}}+1\right)^\nu} \frac{du}{u^2\left(1+\frac{1}{u^2}\right)} \\ &= \int_0^\infty \frac{u^{\mu\nu+1}}{(u^{2\mu}+1)^\nu} \frac{du}{1+u^2}. \end{aligned} \tag{2.2}$$

Averaging (2.1) and (2.2) yields

$$I(\mu,\nu) = \frac{1}{2}\int_0^\infty \frac{u^{\mu\nu-1}+u^{\mu\nu+1}}{(u^{2\mu}+1)^\nu} \frac{du}{1+u^2} = \frac{1}{2}\int_0^\infty \frac{u^{\mu\nu-1}}{(u^{2\mu}+1)^\nu}\, du.$$

Substituting $y = u^{2\mu}$, $du = \frac{dy}{2\mu y^{1-\frac{1}{2\mu}}}$, while the limits of integration remain unchanged, we find

$$\begin{aligned} I(\mu,\nu) &= \frac{1}{2}\int_0^\infty \frac{(y^{\frac{1}{2\mu}})^{\mu\nu-1}}{(y+1)^\nu} \frac{dy}{2\mu y^{1-\frac{1}{2\mu}}} = \frac{1}{4\mu}\int_0^\infty \frac{y^{\frac{\nu}{2}-1}}{(1+y)^\nu}\, dy \\ &= \frac{1}{4\mu}\int_0^\infty \frac{y^{\frac{\nu}{2}-1}}{(1+y)^{\left(\frac{\nu}{2}\right)+\left(\frac{\nu}{2}\right)}}\, dy. \end{aligned} \tag{2.3}$$

Recalling the following integral representation for the beta function of

$$\mathrm{B}(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy, \quad m,n>0,$$

comparing this to (2.3), we see that

$$I(\mu,\nu) = \frac{1}{4\mu}\mathrm{B}\left(\frac{\nu}{2},\frac{\nu}{2}\right), \quad \nu > 0.$$

From the well-known relation between the beta function and the gamma function of

$$\mathrm{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},$$

we have

$$I(\mu,\nu) = \frac{1}{4\mu}\frac{\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu}{2}+\frac{\nu}{2})} = \frac{1}{4\mu}\frac{\Gamma^2(\frac{\nu}{2})}{\Gamma(\nu)}, \quad \mu>0, \nu>0.$$

Consider $\mu < 0$. Let $\mu = -a$ where $a > 0$. Then

$$I(-a,\nu) = \int_0^{\pi/2} \frac{1}{(\tan^{-a}x+\cot^{-a}x)^\nu} \frac{dx}{\tan x}.$$

Again enforcing a substitution of $u = \tan x$, produces

$$I(-a,\nu) = \int_0^\infty \frac{1}{\left(u^{-a}+\frac{1}{u^{-a}}\right)^\nu} \frac{du}{u(1+u^2)} = \int_0^\infty \frac{u^{a\nu-1}}{(u^{2a}+1)^\nu}\frac{du}{1+u^2}.$$

The right most integral is just the integral given in (2.1) with μ replaced with a. Thus

$$I(-a,\nu) = \frac{1}{4a}\frac{\Gamma^2(\frac{\nu}{2})}{\Gamma(\nu)}, \quad a>0, \nu>0.$$

But since $\mu = -a$, we see that

$$I(\mu, \nu) = -\frac{1}{4\mu} \frac{\Gamma^2(\frac{\nu}{2})}{\Gamma(\nu)}, \quad \mu < 0, \nu > 0.$$

In summary, the correct value for the integral appearing under Entry **3.688.14** of Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* (8th ed., 2007) is:

$$\int_0^{\pi/2} \frac{1}{(\tan^\mu x + \cot^\mu x)^\nu} \frac{dx}{\tan x} = \frac{1}{4|\mu|} \frac{\Gamma^2(\frac{\nu}{2})}{\Gamma(\nu)}, \quad \mu \neq 0, \nu > 0.$$

GRE Practice #17: Let G be a group with at most 30 elements and consider its action on itself by conjugation: given $x \in G$ the action of $g \in G$ on x is defined by $g^{-1}xg$. Assume for each x the stabilizer is just $\{e\}$, the identity element. How many such groups are there modulo 5?.

(a) 0 (b) 1 (c) 2 (d) 3 (e) 5.

Solution: A natural proof is to use the Orbit-Stabilizer Counting Theorem. Since each G_{x_i} is just one element, and $X = G$, we find $\#G = \sum_{i=1}^k \#G/\#G_{x_i}$. As each $\#G_{x_i} = 1$ we get

$$\#G = \sum_{i=1}^k \#G/1;$$

the only way this can work is if $k = 1$ (if $k > 1$ then the RHS is larger than the LHS). Thus everything is in the same orbit! Note $\#Gx = \#G/\#G_x$ and the orbit of the identity e is just e (as for all g we have $g^{-1}eg = e$). As the orbit of e is one element, and the Orbit Stabilizer formula gives us $\#G = \#Ge$, we conclude $G = \{e\}$. Thus the answer is (b) 1.

However, while the problem may scream using this theorem, we can solve it directly! It's always worth pausing and seeing if we have to go a certain route; while that may be one way to get to the answer, all we care about is reaching that answer. Clearly $x \in G_x$, the stabilizer of x (as is x^{-1} and more generally any power of x, as $x^{-1}xx = x$). As we are told $G_x = \{e\}$ we have that $x = e$. In other words, $G = \{e\}$!

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